RMO 2003

Exercise 1 (RMO 2003 P2,). If n is an integer greater than 7, prove that $\binom{n}{7} - \left[\frac{n}{7}\right]$ is divisible by 7. [Here $\binom{n}{7}$ denotes the number of ways of choosing 7 objects from among n objects; also for any real number x, $[x]$ denotes the greatest integer not exceeding x.]

Solution 1. If $x_1, x_2, \ldots, x_k$ are real numbers with $k \geq 2$ and $1 \leq \ell \neq k$ is an integer, then we denote by $x_1 \ldots \widehat{x_\ell} \ldots x_k$ the product

$$x_1 \ldots x_{\ell-1} x_{\ell+1} \ldots x_k.$$

Let n be an integer greater than 7. Let q, r be integers with $0 \leq r \leq 6$ and $q \geq 1$ such that n is equal to $7q+r$. We need to show that $\binom{7q+r}{7} - q$ is divisible by 7, which is equivalent to

$$6!\left(\binom{7q+r}{7} - q\right) \equiv 0 \bmod 7 \tag{1}$$

since 6! is coprime to 7. Note that $6!\binom{7q+r}{7}$ is equal to

$$\frac{(7q+r)(7q+r-1)(7q+r-2)(7q+r-3)(7q+r-4)(7q+r-5)(7q+r-6)}{7}.$$

It follows that

$$6!\binom{7q+r}{7} \equiv q \cdot r(r-1)\cdots\widehat{(r-r)}\cdots(r-6) \bmod 7 \equiv q \cdot 6! \bmod 7.$$

This shows that Eq. (1) holds, which proves the result. ■

Exercise 2 (RMO 2003 P3,). Let a, b, c be three positive real numbers such that $a+b+c = 1$. Prove that among the three numbers $a-ab, b-bc, c-ca$, there is one which is at most $\frac{1}{4}$ and there is one which is at least $\frac{2}{9}$.

Solution 2. Note that

$$(a-ab)(b-bc)(c-ca) = a(1-a)b(1-b)c(1-c) \leq \frac{1}{4^3}$$

holds. So among $a-ab, b-bc, c-ca$ there is one which is at most $\frac{1}{4}$.

Also note that

$$\begin{aligned}(a-ab)+(b-bc)+(c-ca) &= 1-(ab+bc+ca)\\ &\geq 1-\frac{1}{3}(a+b+c)^2\\ &\geq \frac{2}{3}\end{aligned}$$

holds. This shows that one of $a-ab, b-bc, c-ca$ is at least $\frac{2}{9}$. ■

Exercise 3 (RMO 2003 P4,). Find the number of ordered triples (x, y, z) of nonnegative integers satisfying the conditions:

$$x \leq y \leq z, x+y+z \leq 100.$$

Solution 3. Substituting $y = x+a, z = y+b$, we get $3x+2a+b \leq 100$. So it suffices to find number of solutions of the inequality $2a+b \leq 100-3x$ in nonnegative integers. Note that given a nonnegative integer k, the number of solutions of the inequality $2u+v=k$ in nonnegative integers is equal to $\left\lfloor \frac{k}{2} \right\rfloor + 1$. Hence given a nonnegative integer k, the number of solutions of $2u+v \leq k$ in nonnegative integers is equal to

$$\sum_{t=0}^{k} \left(\left\lfloor \frac{t}{2} \right\rfloor + 1 \right).$$

Note that if $k = 2\ell+1$, then

$$\begin{aligned}\sum_{t=0}^{k} \left\lfloor \frac{t}{2} \right\rfloor &= \sum_{t=0}^{2\ell+1} \left\lfloor \frac{t}{2} \right\rfloor\\ &= 1+1+2+2+\cdots+\ell+\ell\\ &= 2\left(1+2+\cdots+\ell\right)\\ &= 2\left(1+2+\cdots+\left\lfloor \frac{k}{2} \right\rfloor\right),\end{aligned}$$

and if $k = 2\ell$, then

$$\begin{aligned}\sum_{t=0}^{k} \left\lfloor \frac{t}{2} \right\rfloor &= \sum_{t=0}^{2\ell} \left\lfloor \frac{t}{2} \right\rfloor\\ &= 1+1+2+2+\cdots+(\ell-1)+(\ell-1)+\ell\\ &= 2\left(1+2+\cdots+\ell\right)-\ell\\ &= 2\left(1+2+\cdots+\left\lfloor \frac{k}{2} \right\rfloor\right)-\left\lfloor \frac{k}{2} \right\rfloor.\end{aligned}$$

The content posted here and at this blog by Evan Chen are quite useful.

Thus

$$\sum_{t=0}^{k} \left\lfloor \frac{t}{2} \right\rfloor = 2\left(1+2+\cdots+\left\lfloor \frac{k}{2} \right\rfloor\right) - \frac{1+(-1)^k}{2}\left\lfloor \frac{k}{2} \right\rfloor,$$

and hence given a nonnegative integer k, the number of solutions of $2u+v \le k$ in nonnegative integers is equal to

$$\begin{aligned}\sum_{t=0}^{k}\left(\left\lfloor \frac{t}{2} \right\rfloor + 1\right) &= k+1+\sum_{t=0}^{k}\left\lfloor \frac{t}{2} \right\rfloor \\ &= k+1+2\left(1+2+\cdots+\left\lfloor \frac{k}{2} \right\rfloor\right) - \frac{1+(-1)^k}{2}\left\lfloor \frac{k}{2} \right\rfloor.\end{aligned}$$

So the number of solutions of $2a+b \le 100-3x$ in nonnegative integers is equal to

$$\begin{aligned}&\sum_{x=0}^{33}\sum_{t=0}^{100-3x}\left(\left\lfloor \frac{t}{2} \right\rfloor + 1\right) \\ &= \sum_{x=0}^{33}\left(100-3x+1+2\left(1+2+\cdots+\left\lfloor \frac{100-3x}{2} \right\rfloor\right) - \frac{1+(-1)^{100-3x}}{2}\left\lfloor \frac{100-3x}{2} \right\rfloor\right).\end{aligned}$$

Decomposing this sum over even and odd values of x, we get

$$\begin{aligned}&\sum_{x=0}^{33}\sum_{t=0}^{100-3x}\left(\left\lfloor \frac{t}{2} \right\rfloor + 1\right) \\ &= \sum_{m=0}^{16}\left(100-6m+1+2\left(1+2+\cdots+\left\lfloor \frac{100-6m}{2} \right\rfloor\right) - \left\lfloor \frac{100-6m}{2} \right\rfloor\right) \\ &\qquad + \sum_{m=0}^{16}\left(100-6m-3+1+2\left(1+2+\cdots+\left\lfloor \frac{100-6m-3}{2} \right\rfloor\right)\right) \\ &= \sum_{m=0}^{16}\big((50-3m+1+2(1+2+\cdots+(50-3m))) \\ &\qquad + (100-6m-2+2(1+2+\cdots+(48-3m)))\big) \\ &= \sum_{m=0}^{16}(-(50-3m)+1+4(1+2+\cdots+(50-3m))) \\ &= \sum_{m=0}^{16}(3m-49+2(50-3m)(51-3m)) \\ &= \sum_{m=0}^{16}(18m^2-603m+5051) \\ &= 3\cdot 16\cdot 17\cdot 33 - 603\cdot 8\cdot 17 + 5051\cdot 17\end{aligned}$$

$$= 17(1600 - 16 + 5051 - 4824) = 17 \cdot 1811 = 30787.$$

Consequently, there are 30787 ordered triples (x, y, z) of nonnegative integer satisfying the given conditions. ■

The following is a cleaner and more streamlined solution to the above problem. It is from a post on AoPS by Rijul Saini. See also this page.

Solution 4. Let us establish the following.

Claim — Let m be a nonnegative integer. The number of solutions of

$$p \leq q, p + q \leq m$$

in nonnegative integers is equal to

$$\begin{cases} \frac{(m+1)(m+3)}{4} & \text{if } m \text{ is odd,} \\ \frac{(m+2)^2}{4} & \text{if } m \text{ is even.} \end{cases}$$

Proof of the Claim. Note that for a nonnegative integer k, the number of solutions of

$$x \leq y, x + y = k,$$

in the nonnegative integers is equal to

$$\begin{cases} \frac{k+1}{2} & \text{if } k \text{ is odd,} \\ \frac{k+2}{2} & \text{if } k \text{ is even.} \end{cases}$$

Also note that the Claim follows for $m = 0$.

If m is an odd positive integer, then the number of solutions of

$$x \leq y, x + y \leq m$$

in the nonnegative integers is equal to

$$1 + \frac{1+1}{2} + \frac{2+2}{2} + \frac{3+1}{2} + \frac{4+2}{2} + \frac{5+1}{2} + \cdots + \frac{(m-1)+2}{2} + \frac{m+1}{2}$$
$$= 1 + 1 + 2 + 2 + 3 + 3 + \cdots + \frac{m+1}{2} + \frac{m+1}{2}$$
$$= \frac{m+1}{2}\frac{m+3}{2}.$$

It follows from the above that if m is an even positive integer, then the number of solutions of

$$x \leq y, x + y \leq m$$

in the nonnegativeg integers is equal to

$$\frac{(m-1)+1}{2}\frac{(m-1)+3}{2} + \frac{m+2}{2} = \frac{(m+2)^2}{4}.$$

This proves the Claim. □

Note that $x+y+z \leq 100$ is equivalent to $(y-x)+(z-x) \leq 100-3x$, and if $x \leq y \leq z$, then $y-x \geq 0$ and $z-x \geq 0$. It follows that the solutions of the given equation are in one-to-one correspondence with the solutions of the equation $Y+Z \leq 100-3x$ as x runs from 0 to 33. More precisely, the map

$$(x,y,z) \mapsto (y-x, z-x)$$

defines a bijection between the set

$$\{(x,y,z) \,|\, x,y,z \in \mathbb{Z}_{\geq 0}, x \leq y \leq z, x+y+z \leq 100\}$$

and the set

$$\bigcup_{0 \leq x \leq 33} \{(Y,Z) \,|\, Y,Z \in \mathbb{Z}_{\geq 0}, Y \leq Z, Y+Z \leq 100-3x\}\,.$$

By the above claim, the number of solutions of the given equation is

$$\begin{aligned}
&51^2 + 49 \cdot 50 + 48^2 + 46 \cdot 47 + 45^2 + 43 \cdot 44 + \cdots + 3^2 + 1 \cdot 2 \\
&= \sum_{k=0}^{16} (3k+1)(3k+2) + \sum_{k=1}^{17} (3k)^2 \\
&= \sum_{k=0}^{16} (9k^2+9k+2) + \sum_{k=1}^{17} 9k^2 \\
&= 9 \times 17^2 + 2 \times 9 \times \frac{16 \cdot 17 \cdot 33}{6} + 9 \times \frac{16 \times 17}{2} + 2 \times 17 \\
&= 9 \times 17\,(17 + 176 + 8) + 34 \\
&= 153 \times 201 + 34 \\
&= 30600 + 187 \\
&= 30787.
\end{aligned}$$

■

Exercise 4 (RMO 2003 P6,). Find all real numbers a for which the equation

$$x^2 + (a-2)x + 1 = 3|x|$$

has exactly three distinct real solutions for x.

Solution 5. Let a be a real number such that $x^2 + (a-2)x + 1 = 3|x|$ has exactly three distinct real solutions. Note that the equation

$$(x^2 + (a-2)x + 1 - 3x)(x^2 + (a-2)x + 1 + 3x) = 0$$

also has exactly three distinct real solutions. It follows that the discriminant of one of the polynomials $x^2 + (a-2)x + 1 - 3x$, $x^2 + (a-2)x + 1 + 3x$

vanishes, and the discriminant of the other is positive. The discriminants of these polynomials are

$$\begin{aligned}(a-5)^2-4 &= a^2-10a+21\\ &= (a-3)(a-7),\\ (a+1)^2-4 &= a^2+2a-3\\ &= (a+3)(a-1)\end{aligned}$$

respectively. It follows that a does not belong to $(-\infty,-3)\cup(-3,1)\cup(1,3)\cup(3,7)\cup(7,\infty)$, or equivalently, a belongs to $\{-3,1,3,7\}$.

Let us determine whether the given equation has three distinct real roots if a lies in $\{-3,1,3,7\}$. Let us assume that a lies in $\{-3,1,3,7\}$. Note that then one of $(a-3)(a-7),(a+3)(a-1)$ vanishes and another is positive, and consequently, one of the polynomials

$$x^2+(a-2)x+1-3x, x^2+(a-2)x+1+3x$$

have distinct real roots, and the other has a double root, which is a real number. Observe that if $x^2+(a-2)x+1-3x$ has a double root, then that root is equal to $-\frac{a-5}{2}$. Note that

$$\begin{aligned}\left(\frac{a-5}{2}\right)^2-(a+1)\left(\frac{a-5}{2}\right)+1 &= \frac{1}{4}\left(a^2-10a+25-2a^2+8a+10+4\right)\\ &= \frac{1}{4}\left(-a^2-2a+39\right),\end{aligned}$$

which has a negative discriminant. This shows that if $x^2+(a-2)x+1-3x$ has a real double root, then that cannot be a zero of $x^2+(a-2)x+1+3x$. Also note that

$$\begin{aligned}\left(\frac{a+1}{2}\right)^2-(a-5)\frac{a+1}{2}+1 &= \frac{1}{4}\left(a^2+2a+1-2a^2+8a+10+4\right)\\ &= \frac{1}{4}\left(-a^2+10a+15\right),\end{aligned}$$

whose roots are not integers. Using that a is an integer, it follows that if $x^2+(a-2)x+1+3x$ has a real double root, then that cannot be a root of $x^2+(a-2)x+1-3x$. We conclude that if a lies in $\{-3,1,3,7\}$, then

$$(x^2+(a-2)x+1-3x)(x^2+(a-2)x+1+3x)=0$$

has exactly three distinct real solutions, or equivalently, the equation

$$x^2+(a-2)x+1=3|x|$$

has exactly three distinct real solutions for x.

So the required real numbers are $a=-3,1,3,7$. ■

Exercise 5 (RMO 2003 P7,). Consider the set $X = \{1, 2, 3, \ldots, 9, 10\}$. Find two disjoint nonempty subsets A and B of X such that

(a) $A \cup B = X$,

(b) $\text{prod}(A)$ is divisible by $\text{prod}(B)$, where for any finite set of numbers C, $\text{prod}(C)$ denotes the product of all numbers in C,

(c) the quotient $\text{prod}(A)/\text{prod}(B)$ is as small as possible.

Summary — It is equivalent to finding a subset B of $\{1, \ldots, 10\}$, other than $\emptyset, \{1, \ldots, 10\}$, such that $\text{prod}(B)^2$ divides $10!$ and the quotient $10!/\text{prod}(B)^2$ is minimized. To do so,

(a) write down the prime power factorization of $10!$,

(b) throw in enough elements in B so that $\text{prod}(B)$ is maximized, and $\text{prod}(B)^2$ divides $10!$.

Walkthrough —

(a) Observe that it is enough to find a nonempty proper subset B of $\{1, 2, \ldots, 10\}$ such that $\text{prod}(B)^2$ divides $10!$ and $\text{prod}(B)$ is the maximum.

(b) Writing down the prime power factorization of $10!$, deduce that B does not contain 7, it contains a multiple of 5, and also a multiple of 2 and a multiple of 3.

(c) Prove that B contains exactly one multiple of 5, and not more that two multiples of 3.

(d) Show that B is equal to one of the subsets $\{5, 3, 6, 2^3\}$, $\{5, 3, 6, 2^3, 1\}$, $\{5, 3, 6, 2, 2^2\}$, $\{5, 3, 6, 2, 2^2, 1\}$, $\{5, 9, 2, 2^3\}$, $\{5, 9, 2, 2^3, 1\}$, $\{10, 3, 6, 2^2\}$, $\{10, 3, 6, 2^2, 1\}$, $\{10, 9, 2^3\}$, $\{10, 9, 2^3, 1\}$, $\{10, 9, 2^3\}$, $\{10, 9, 2^3, 1\}$.

(e) Show that any of these three subsets also have the stated property.

First, let's work on it. Let A, B be two nonempty disjoint subsets of X satisfying the required conditions (note that such subsets exist since X can be written as the union of two disjoint subsets in finitely many ways only). Due to the equality

$$\frac{\text{prod}(A)}{\text{prod}(B)} = \frac{10!}{(\text{prod}(B))^2},$$

it is equivalent to having a subset B of X such that $\text{prod}(B)^2$ divides $10!$ and $\text{prod}(B)$ is the maximum. Note that $10!$ is equal to the product $2^8 \cdot 3^4 \cdot 5^2 \cdot 7$. So $\text{prod}(B)$ divides $2^4 \cdot 3^2 \cdot 5$, and hence, B does not contain 7. Moreover, B contains a multiple of 5, otherwise $(\text{prod}(B \cup \{5\}))^2$ would divide $10!$ and $\text{prod}(B \cup \{5\})$ would be strictly larger than $\text{prod}(B)$, which contradicts the choice of B. Similarly, B also contains a multiple of 2 and a multiple of 3. Note

that B contains exactly one multiple of 5 (since $5^3 \nmid 10!$). Since $(\text{prod}(B))^2$ divides 10! and $\text{prod}(B)$ is the maximum, B is equal to one of the following sets

- $\{5,3,2,2^3\}, \{5,3,2,2^3,1\}, \{5,6,2,2^3\}, \{5,6,2,2^3,1\}, \{5,3,6,2^3\}, \{5,3,6,2^3,1\}, \{5,3,6,2,2^2\}, \{5,3,6,2,2^2,1\}, \{5,9,2,2^3\}, \{5,9,2,2^3,1\}$ if B contains 5,
- $\{10,3,2^3\}, \{10,3,2^3,1\}, \{10,3,2,2^2\}, \{10,3,2,2^2,1\}, \{10,6,2^2\}, \{10,6,2^2,1\}, \{10,3,6,2^2\}, \{10,3,6,2^2,1\}, \{10,9,2^3\}, \{10,9,2^3,1\}, \{10,9,2,2^2\}, \{10,9,2,2^2,1\}$ if B contains 10.

For any of the above sets, the product of its elements is equal to 240, 480, or 720. So B is equal to one of the sets $\{5,3,6,2^3\}, \{5,3,6,2^3,1\}, \{5,3,6,2,2^2\}, \{5,3,6,2,2^2,1\}, \{5,9,2,2^3\}, \{5,9,2,2^3,1\}, \{10,3,6,2^2\}, \{10,3,6,2^2,1\}, \{10,9,2^3\}, \{10,9,2^3,1\}, \{10,9,2^3\}, \{10,9,2^3,1\}$.

Also note that if B denotes one of these subsets of $\{1,\dots,10\}$, then $\text{prod}(B)^2$ divides 10! and $\text{prod}(B)$ is the maximum.

This proves that $\{5,3,6,2^3\}, \{5,3,6,2^3,1\}, \{5,3,6,2,2^2\}, \{5,3,6,2,2^2,1\}, \{5,9,2,2^3\}, \{5,9,2,2^3,1\}, \{10,3,6,2^2\}, \{10,3,6,2^2,1\}, \{10,9,2^3\}, \{10,9,2^3,1\}, \{10,9,2^3\}, \{10,9,2^3,1\}$ are precisely all the subsets of $\{1,\dots,10\}$ having the required property. Thus we could take $A=\{1,2,3,4,5,6,7\}, B=\{8,9,10\}$ for instance. ♣

> **Remark.** Note that the above discussion provides more than what has been required. After observing that $\text{prod}(B)$ divides $2^4\cdot 3^2\cdot 5$, one may show that there is a subset B with $\text{prod}(B)$ equal to $2^4\cdot 3^2\cdot 5$ (for instance, $B=\{8,9,10\}$), and then conclude.

Solution 6. Let A, B be two nonempty disjoint subsets of X satisfying the required conditions (note that such subsets exist since X can be written as the union of two disjoint subsets in finitely many ways only). Due to the equality

$$\frac{\text{prod}(A)}{\text{prod}(B)} = \frac{10!}{(\text{prod}(B))^2},$$

it is equivalent to having a subset B of X such that $\text{prod}(B)^2$ divides 10! and $\text{prod}(B)$ is the maximum. Note that 10! is equal to the product $2^8\cdot 3^4\cdot 5^2\cdot 7$. So $\text{prod}(B)$ divides $2^4\cdot 3^2\cdot 5$. If $B=\{8,9,10\}$, then $\text{prod}(B)$ is equal to $2^4\cdot 3^2\cdot 5$. Hence, $A=\{1,\dots,7\}, B=\{8,9,10\}$ are two disjoint nonempty subsets of $X=\{1,\dots,10\}$ satisfying the required conditions. ■

> **Remark.** Don't be surprised that it took a bit long to arrive at the above solution. It is often the case. Further, it is a standard practice to write down a complete solution as the final one, without any reference to the prior attempts (possibly several). Those attempts have their important role in providing insights, which may lead to a solution. Here, the details of those attempts have

> not been hidden from you, in order to take you along the journey. However, I would like to highlight that a *solution* to a problem has to be complete, and at the same time, has to be free from the prior thoughts that have no direct role to play in that solution, though they might have played a significant role in gaining insight.