# RMO 2002

**Exercise 1** (RMO 2002 P2, ). Solve the following equation for real $x$:

$$(x^2+x-2)^3+(2x^2-x-1)^3=27(x^2-1)^3.$$

**Solution 1.** The given equation is equivalent to

$$(x^2+x-2)^3+(2x^2-x-1)^3+(-3x^2+3)^3=0.$$

Note that $x^2+x-2, 2x^2-x-1, -3x^2+3$ add up to zero. This implies

$$\begin{aligned}&(x^2+x-2)^3+(2x^2-x-1)^3+(-3x^2+3)^3\\&=3(x^2+x-2)(2x^2-x-1)(-3x^2+3)\\&=-9(x+2)(x-1)(x-1)(2x-1)(x-1)(x+1).\end{aligned}$$

Thus the required solutions for $x$ are

$$-2,-1,\frac{1}{2},1.$$

■

**Exercise 2** (RMO 2002 P3, , cf. RMO 2012d P2, RMO 2012a P2, RMO 2012b P2, RMO 2012c P2). Let $a,b,c$ be positive integers such that $a$ divides $b^2$, $b$ divides $c^2$ and $c$ divides $a^2$. Prove that $abc$ divides $(a+b+c)^7$.

**Solution 2.** In the following, $k$ denotes one of the integers $2,3,4,5$. Assume that $a \mid b^k, b \mid c^k, c \mid a^k$. Note that if one of $a,b,c$ is equal to 1, then all of them are equal to 1, and then there is nothing to prove. Henceforth, let us assume that $a,b,c$ are larger than 1. The given conditions imply that the set of prime divisors of $a,b,c$ are equal.

Let $p$ be a common prime divisor of $a,b,c$. Let $p^\alpha, p^\beta, p^\gamma$ denote the highest powers of $p$ dividing $a,b,c$ respectively. The given divisibility conditions imply that

$$\alpha \le k\beta, \beta \le k\gamma, \gamma \le k\alpha.$$

This yields

$$\alpha+\beta+\gamma \le \alpha + k^2\alpha + 2\alpha = (k^2+k+1)\alpha,$$

and similarly, $\alpha+\beta+\gamma \le (k^2+k+1)\beta$, $\alpha+\beta+\gamma \le (k^2+k+1)\gamma$, which shows that

$$\alpha+\beta+\gamma \le (k^2+k+1)\min\{\alpha,\beta,\gamma\}.$$

Note that $p^{\min\{\alpha,\beta,\gamma\}}$ divides $a+b+c$. By the above inequality, $p^{\alpha+\beta+\gamma}$ divides $(a+b+c)^{k^2+k+1}$. So $abc$ divides $(a+b+c)^{k^2+k+1}$. Also note that $p^{\alpha+\beta+\gamma}$ is the highest power of $p$ dividing $abc$. This shows that for each common prime divisor $p$ of $a,b,c$, the highest power of $p$ that divides $abc$ also divides $(a+b+c)^{k^2+k+1}$. It follows that $abc$ divides $(a+b+c)^{k^2+k+1}$. ■

**Solution 3.** It suffices to show that $abc$ divides $a^\alpha b^\beta c^\gamma$ for any nonnegative integers $\alpha,\beta,\gamma$ such that $\alpha+\beta+\gamma = k^2+k+1$. Note that

$$abc \mid ac^{k+1} \mid a^{k^2+k+1}, \quad abc \mid a^{k+1}b \mid b^{k^2+k+1}, \quad abc \mid b^{k+1}c \mid c^{k^2+k+1},$$

which shows that $abc$ divides $a^{k^2+k+1}, b^{k^2+k+1}, c^{k^2+k+1}$. Moreover, $abc$ divides $a^\alpha b^\beta c^\gamma$ when $\alpha,\beta,\gamma$ are all positive. It remains to consider the case when exactly one of $\alpha,\beta,\gamma$ is zero, which we assume from now on. If $\alpha = 0$, then

$$abc \mid b^{k+1}c \mid b^\beta c^\gamma \text{ if } \beta \ge k+1,$$
$$abc \mid c^{k^2+1}b \mid b^\beta c^\gamma \text{ if } \beta \le k$$

holds. If $\beta = 0$, then

$$abc \mid c^{k+1}a \mid c^\gamma a^\alpha \text{ if } \gamma \ge k+1,$$
$$abc \mid a^{k^2+1}c \mid c^\gamma a^\alpha \text{ if } \gamma \le k$$

holds. If $\gamma = 0$, then

$$abc \mid a^{k+1}b \mid a^\alpha b^\beta \text{ if } \alpha \ge k+1,$$
$$abc \mid b^{k^2+1}a \mid a^\alpha b^\beta \text{ if } \alpha \le k$$

holds. This shows that $abc$ divides $(a+b+c)^{k^2+k+1}$. ■

**Exercise 3** (RMO 2002 P4, ). Suppose the integers $1,2,3,\ldots,10$ are split into two disjoint collections $a_1,a_2,a_3,a_4,a_5$ and $b_1,b_2,b_3,b_4,b_5$ such that $a_1<a_2<a_3<a_4<a_5$ and $b_1>b_2>b_3>b_4>b_5$.

(i) Show that the larger number in any pair $\{a_j,b_j\}$, $1\le j\le 5$, is at least 6.

(ii) Show that $|a_1-b_1|+|a_2-b_2|+|a_3-b_3|+|a_4-b_4|+|a_5-b_5| = 25$ for every such partition.



The content posted here and at this blog by Evan Chen are quite useful.

**Solution 4.** Let $1 \leq j \leq 5$ be an integer. Note that the inequalities

$$a_1 < a_2 < \cdots < a_j, \quad b_j > b_{j+1} > \cdots > b_5$$

hold. Since $a_1, a_2, \ldots, a_j, b_j, b_{j+1}, \ldots, b_5$ are six distincts elements of the set $\{1, 2, \ldots, 10\}$, their maximum is at least 6. The above inequalities show that this maximum is equal to $\max\{a_j, b_j\}$. This proves part (i).

Note that the sets $\{a_1, b_1\}, \{a_2, b_2\}, \{a_3, b_3\}, \{a_4, b_4\}, \{a_5, b_5\}$ are pairwise disjoint subsets of $\{1, 2, \ldots, 10\}$, and the maximum of any of these sets is at least 6. Consequently, the maximums of these sets are equal to $6, 7, 8, 9, 10$ in some order. Thus their minimums are equal to $1, 2, 3, 4, 5$ in some order. Hence, the integer

$$|a_1 - b_1| + |a_2 - b_2| + |a_3 - b_3| + |a_4 - b_4| + |a_5 - b_5|$$

is equal to the difference of the sum of of the maximums of these sets and the sum of the minimums of these sets, which is equal to

$$6 + 7 + 8 + 9 + 10 - (1 + 2 + 3 + 4 + 5) = 25.$$

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**Example 4.** For any natural number $n > 1$, prove the inequality

$$\frac{1}{2} < \frac{1}{n^2+1} + \frac{2}{n^2+2} + \frac{3}{n^2+3} + \cdots + \frac{n}{n^2+n} < \frac{1}{2} + \frac{1}{2n}.$$

**Solution 5.** Note that

$$\begin{aligned}
&\frac{1}{n^2+1} + \frac{2}{n^2+2} + \frac{3}{n^2+3} + \cdots + \frac{n}{n^2+n} \\
&> \frac{1}{n^2+n}(1 + 2 + 3 + \cdots + n) \quad (\text{using } n > 1) \\
&= \frac{1}{2}.
\end{aligned}$$

Also note that

$$\begin{aligned}
&\frac{1}{n^2+1} + \frac{2}{n^2+2} + \frac{3}{n^2+3} + \cdots + \frac{n}{n^2+n} \\
&< \frac{1}{n^2+1}(1 + 2 + 3 + \cdots + n) \quad (\text{using } n > 1) \\
&= \frac{n^2+n}{2(n^2+1)} \\
&= \frac{1}{2} + \frac{n-1}{2(n^2+1)} \\
&< \frac{1}{2} + \frac{1}{2n}.
\end{aligned}$$

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**Exercise 5** (RMO 2002 P7, ). Find all integers $a, b, c, d$ satisfying the following relations:

1. $1 \leq a \leq b \leq c \leq d$;
2. $ab + cd = a + b + c + d + 3$.

**Solution 6.** Let $a, b, c, d$ be integers satisfying the given conditions. Let $p, q, r$ be integers satisfying $b = a + p, c = b + q, d = c + r$. Note that $p, q, r$ are nonnegative. The given conditions yield

$$a(a + p) + (a + p + q)(a + p + q + r) = 4a + 3p + 2q + r + 3,$$

which is equivalent to

$$2a^2 + (3p + 2q + r)a + (p + q)(p + q + r) = 4a + 3p + 2q + r + 3 = 0,$$

that is,

$$2a(a - 2) + (a - 1)(3p + 2q + r) + (p + q)(p + q + r) = 3. \tag{1}$$

From Eq. (1), we get $a \leq 2$.

If $a = 1$ holds, then $(p + q)(p + q + r) = 5$, which gives

$$p + q = 1, p + q + r = 5,$$

that is, $(p, q, r)$ is equal to $(1, 0, 4)$ or $(0, 1, 4)$.

Let us consider the case that $a = 2$. Note that

$$(3p + 2q + r) + (p + q)(p + q + r) = 3,$$

which implies that $p = 0$. It follows that $2q + r + q(q + r) = 3$, which gives $q \leq 1$, and hence $(q, r)$ is equal to $(0, 3)$ or $(1, 0)$.

The above argument shows that $(p, q, r)$ is equal to one of

$$(1, 0, 4), (0, 1, 4), (0, 0, 3), (0, 1, 0).$$

Consequently, $(a, b, c, d)$ is equal to one of

$$(1, 2, 2, 6), (1, 1, 2, 6), (2, 2, 2, 5), (2, 2, 3, 3).$$

Moreover, the above tuples also satisfy the given conditions.

We conclude that the above tuples are precisely all the solutions to the given relations. ■