RMO 2001

Exercise 1 (RMO 2001 P2,). Find all primes p, q such that $p^2 + 7pq + q^2$ is a perfect square.

Solution 1. Let p, q be primes such that

$$p^2 + 7pq + q^2 = m^2 \tag{1}$$

holds for some positive integer m. Note that m is congruent to one of $q, -q$ modulo p. Write $m = kp \pm q$ for some integer k. Substituting $m = kp \pm q$ in Eq. (1) yields

$$p(k^2 - 1) = (7 \mp 2k)q. \tag{2}$$

Let us consider the case that $p > q$. This gives $m^2 \geq 9q^2$, and hence we obtain $k > 1$. This implies that $k^2 - 1$ is positive, and hence, so is $7 \pm 2k$. Using $p > q$, we get

$$k^2 - 1 < 7 \mp 2k,$$

which yields $(k \pm 1)^2 < 9$. Noting that k is positive, it follows that $k \pm 1$ is equal to one of $0, 1, 2$, and hence k is equal to one of $2, 3$. Substituting $k = 2$ in Eq. (2) yields $3p = (7 \mp 4)q$. Since $p > q$, we obtain $3p = 11q$, which shows that $p = 11, q = 3$. Substituting $k = 3$ in Eq. (2) yields $8p = (7 \mp 6)q$, which implies that $8p = q$ or $8p = 13q$, which is impossible. It follows that any pair of primes (p, q) such that $p^2 + 7pq + q^2$ is a perfect square and $p > q$ holds, is equal to $(11, 3)$. Note that

$$11^2 + 7 \cdot 11 \cdot 3 + 3^2 = 121 + 231 + 9 = 361 = 19^2$$

holds. We conclude that $(p, q) = (11, 3)$ is the only solution when $p > q$.

Since $p^2 + 7pq + q^2$ is symmetric in p, q, it follows that $(p, q) = (3, 11)$ is the only solution when $p < q$.

Also note that if $p = q$, then $p^2 + 7pq + q^2 = (3p)^2$ is a perfect square. So the solutions are are precisely $(3, 11), (11, 3)$, and the tuples of the form (r, r), where r runs over the set of primes. ■

Exercise 2 (RMO 2001 P3,). Find the number of positive integers x which satisfy the condition

$$\left[\frac{x}{99}\right] = \left[\frac{x}{101}\right].$$

(Here $[z]$ denotes, for any real z, the largest integer not exceeding z, eg, $[\frac{7}{4}] = 1$).

Solution 2. Let x be a positive integer satisfying the given equation. Note that

$$\frac{x}{99} - \frac{x}{101} < 1$$

holds. Indeed, if a, b are real numbers satisfying $a - b \geq 1$, then

$$a \geq b + 1 \geq [b] + 1,$$

which yields $[a] \geq [b] + 1$. It follows that

$$x \leq \frac{99 \cdot 101}{2}.$$

Note that we have the following increasing sequence of integers

$$\begin{aligned} &0 < 99 < 101 < 2 \cdot 99 < 2 \cdot 101 < 3 \cdot 99 < 3 \cdot 101 \\ &< \cdots \\ &< 49 \cdot 99 < 49 \cdot 101 < 50 \cdot 99 < 50 \cdot 101. \end{aligned}$$

since $n \cdot 101 < (n+1) \cdot 99$ holds for a positive integer n if $n < \frac{99}{2}$. Using $\frac{99 \cdot 101}{2} < 51 \cdot 99$, it follows that the positive integers satisfying the given equation lie strictly between 0 and $50 \cdot 101$.

Note that the given equation has no solution in the intervals

$$[99, 101), [2{\cdot}99, 2{\cdot}101), [3{\cdot}99, 3{\cdot}101), [4{\cdot}99, 4{\cdot}101), \ldots, [49{\cdot}99, 49{\cdot}101), [50{\cdot}99, 50{\cdot}101).$$

Also note that any element of the intervals

$$[1, 99), [101, 2{\cdot}99), [2{\cdot}101, 3{\cdot}99), [3{\cdot}101, 4{\cdot}99), \cdots, [48{\cdot}101, 49{\cdot}99), [49{\cdot}101, 50{\cdot}99)$$

is a solution to the given equation.

Hence, the number of solutions of the given equation in positive integers is equal to

$$\begin{aligned} &98 + (2 \cdot 99 - 101) + (3 \cdot 99 - 2 \cdot 101) + (4 \cdot 99 - 3 \cdot 101) \\ &\quad + \cdots + (49 \cdot 99 - 48 \cdot 101) + (50 \cdot 99 - 49 \cdot 101) \\ &= 98 - 99 + 99(1 + 2 + 3 + 4 + \cdots + 49 + 50) - 101(1 + 2 + 3 + \cdots + 48 + 49) \\ &= -1 + 99 \cdot 25 \cdot 51 - 101 \cdot 49 \cdot 25 \\ &= -1 + 25(99 \cdot 51 - 99 \cdot 49 - 2 \cdot 49) \\ &= 2499. \end{aligned}$$

■

The content posted here and at this blog by Evan Chen are quite useful.

Exercise 3 (RMO 2001 P4,). Consider an $n \times n$ array of numbers

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{pmatrix}.$$

Suppose each row consists of the n numbers $1, 2, \dots, n$ in some order and $a_{ij} = a_{ji}$ for $i = 1, 2, \dots, n$ and $j = 1, 2, \dots, n$. If n is odd, prove that the numbers $a_{11}, a_{22}, a_{33}, \dots, a_{nn}$ are $1, 2, \dots, n$ in some order.

Solution 3. Let $1 \leq k \leq n$ be an integer. Since a_{ij} is equal to a_{ji} for all $1 \leq i, j \leq n$, the number of occurences of k outside the diagonal is even. Since k appears in each row exactly once, there are total n many k's in this array. So the number of occurences of k on the diagonal differs from n by an even number. Since n is odd, it follows that k appears at least once on the diagonal, which holds for all $1 \leq k \leq n$. Hence, the numbers $a_{11}, a_{22}, a_{33}, \dots, a_{nn}$ are equal to $1, 2, \dots, n$ in some order. ■

Exercise 4 (RMO 2001 P6,). If x, y, z are the sides of a triangle, then prove that

$$|x^2(y-z) + y^2(z-x) + z^2(x-y)| < xyz.$$

Solution 4. Note that

$$\begin{aligned} &x^2(y-z) + y^2(z-x) + z^2(x-y) \\ &= x^2y - zx^2 + y^2z - xy^2 + z^2x - yz^2 \\ &= (x^2 - z^2)y + y^2(z-x) + zx(z-x) \\ &= (z-x)(y^2 + zx - y(z+x)) \\ &= (z-x)(y-z)(y-x). \end{aligned}$$

Since

$$|x^2(y-z) + y^2(z-x) + z^2(x-y)|, xyz$$

are symmetric in x, y, z, without loss of generality, we may (and do) assume that $x \leq y \leq z$. Note that $|(z-x)(y-z)(y-x)| < xyz$ is immediate if any two of x, y, z are equal. It remains to consider the case that x, y, z are distinct, which we assume from now on. Let us use Ravi substitution, that is, let a, b, c be real numbers satisfying

$$x = b + c, y = c + a, z = a + b,$$

where a, b, c are positive by the triangle inequality. Using $x < y < z$, we obtain $a > b > c$. Note that

$$|(z-x)(y-z)(y-x)| = (a-b)(b-c)(a-c)$$

$$\begin{aligned} &< aba \\ &< (a+c)(b+c)(a+b) \\ &= xyz, \end{aligned}$$

where the first inequality follows since $0 < a-b < a, 0 < b-c < b, 0 < a-c < c$ holds. ■

Exercise 5 (RMO 2001 P7,). Prove that the product of the first 1000 positive even integers differs from the product of the first 1000 odd integers by a multiple of 2001.

Solution 5. Note that

$$\begin{aligned} &2 \cdot 4 \cdot 6 \cdot \cdots 1998 \cdot 2000 \\ &\quad - 1 \cdot 3 \cdot 5 \cdot \cdots \cdot 1997 \cdot 1999 \\ &= (2001-1)(2001-3) \cdots (2001-1997)(2001-1999) \\ &\quad - (2001-2)(2001-4) \cdots (2001-1998)(2001-2000) \end{aligned}$$

holds, which yields

$$\begin{aligned} &2 \cdot 4 \cdot 6 \cdots 1998 \cdot 2000 - 1 \cdot 3 \cdot 5 \cdots 1997 \cdot 1999 \\ &\equiv 1 \cdot 3 \cdot 5 \cdots 1997 \cdot 1999 - 2 \cdot 4 \cdot 6 \cdots 1998 \cdot 2000 \bmod 2001. \end{aligned}$$

It follows that

$$2(2 \cdot 4 \cdot 6 \cdots 1998 \cdot 2000 - 1 \cdot 3 \cdot 5 \cdots 1997 \cdot 1999) \equiv 0 \bmod 2001,$$

which implies that

$$2000(2 \cdot 4 \cdot 6 \cdots 1998 \cdot 2000 - 1 \cdot 3 \cdot 5 \cdots 1997 \cdot 1999) \equiv 0 \bmod 2001.$$

This shows that

$$2 \cdot 4 \cdot 6 \cdots 1998 \cdot 2000 - 1 \cdot 3 \cdot 5 \cdots 1997 \cdot 1999 \equiv 0 \bmod 2001,$$

completing the proof. ■