RMO 2000

Exercise 1 (RMO 2000 P2,). Solve the equation $y^3 = x^3 + 8x^2 - 6x + 8$, for positive integers x and y.

Solution 1. Let x, y be positive integers satisfying the given equation. Since x is a positive integer, it follows that $8x^2 - 6x + 8$ is positive, and hence, $y^3 \geq x^3$ holds. This shows that $y = x + k$ for some positive integer k. Substituting $y = x + k$ in the given equation and simplifying, we obtain

$$(3k - 8)x^2 + (3k^2 + 6)x + k^3 - 8 = 0.$$

Since x is positive, it follows that $k \leq 2$. If $k = 1$, then

$$5x^2 - 9x + 7 = 0$$

holds, which shows that $7 \equiv x(x + 1) \pmod 2$, which yields $7 \equiv 0 \pmod 2$, which is impossible. This implies that $k = 2$. It follows that

$$2x^2 - 18x = 0,$$

which gives $x = 9$, and consequently, we obtain $y = 9 + 2 = 11$.

Also note that if $(k, x) = (2, 9)$, then

$$(3k - 8)x^2 + (3k^2 + 6)x + k^3 - 8 = 0$$

holds, and it gives

$$(x + k)^3 = x^3 + 8x^2 - 6x + 8,$$

which shows that $(x, y) = (9, 11)$ is a solution to the given equation.

It follows that $(9, 11)$ is the only solution of the given equation in the positive integers. ■

> **Remark.** After arriving at the above solution, one can rewrite it to make it brief by observing that
>
> $$x^3 + 8x^2 - 6x + 8 - (x + 1)^3 = 5x^2 - 9x + 7,$$
>
> which is positive for any positive integer x (in fact, it is positive for any real

number x), and then concluding that $y = x + k$ for some integer $k > 1$.

Example 2. Suppose $(x_1, x_2, \cdots, x_n, \cdots)$ is a sequence of positive real numbers such that $x_1 \geq x_2 \geq x_3 \geq \cdots x_n \geq \cdots$, and for all n,

$$\frac{x_1}{1} + \frac{x_4}{2} + \frac{x_9}{3} + \cdots + \frac{x_{n^2}}{n} \leq 1.$$

Show that for all $k \geq 1$ the following inequality is satisfied:

$$\frac{x_1}{1} + \frac{x_2}{2} + \frac{x_3}{3} + \cdots + \frac{x_k}{k} \leq 3.$$

Solution 2. Note that for any integer $n \geq 1$, we have

$$\begin{aligned}
&\left(\frac{x_1}{1} + \frac{x_2}{2} + \frac{x_3}{3}\right) \\
&+ \left(\frac{x_4}{4} + \frac{x_5}{5} + \frac{x_6}{6} + \frac{x_7}{7} + \frac{x_8}{8}\right) \\
&\quad + \cdots + \left(\frac{x_{n^2}}{n^2} + \frac{x_{n^2+1}}{n^2+1} + \cdots + \frac{x_{(n+1)^2-1}}{(n+1)^2-1}\right) \\
&\leq \left(\frac{x_1}{1} + \frac{x_1}{1} + \frac{x_1}{1}\right) + \left(\frac{x_4}{4} + \frac{x_4}{4} + \frac{x_4}{4} + \frac{x_4}{4} + \frac{x_4}{4}\right) \\
&+ \cdots + \underbrace{\left(\frac{x_{n^2}}{n^2} + \frac{x_{n^2}}{n^2} + \cdots + \frac{x_{n^2}}{n^2}\right)}_{(2n+1) \text{ terms}} \\
&= (2 \cdot 1 + 1)\frac{x_1}{1} + (2 \cdot 2 + 1)\frac{x_4}{4} + \cdots + (2n+1)\frac{x_{n^2}}{n^2} \\
&\leq (3 \cdot 1)\frac{x_1}{1} + (3 \cdot 2)\frac{x_4}{4} + \cdots + (3n)\frac{x_{n^2}}{n^2} \\
&= 3\left(\frac{x_1}{1} + \frac{x_4}{2} + \cdots + \frac{x_{n^2}}{n}\right) \\
&\leq 3.
\end{aligned}$$

Since for any $k \geq 1$, there is a positive integer m with $(m+1)^2 - 1 \geq k$, the result follows. ■

Exercise 3 (RMO 2000 P4,). All the 7-digit numbers containing each of the digits $1, 2, 3, 4, 5, 6, 7$ exactly once, and not divisible by 5, are arranged in increasing order. Find the 2000-th number in this list.

Solution 3. In this list, first $6! - 5! = 600$ numbers has 1 as their first digit. Similarly, the next 600 numbers begin with 2 and next to them, there are exactly 600 numbers that begin with 3. The next $5! - 4! = 96$ numbers begin with 41, and the next $5! - 4! = 96$ numbers begin with 42. Note that $2000 - 600 - 600 - 600 - 96 - 96 = 8$, and hence we need to find the 8-th

The content posted here and at this blog by Evan Chen are quite useful.

among the numbers in this list that begin with 431. Among them, there are $3! - 2! = 4$ numbers that begin with $4, 3, 1, 2$. The next few numbers are $4315267, 4315276, 4315627, 4315672, 4315726, 4315762$. Hence, 2000th number in this list is 4315672. ■

Exercise 4 (RMO 2000 P6,).

1. Consider two positive integers a and b which are such that $a^a b^b$ is divisible by 2000. What is the least possible value of the product ab?

2. Consider two positive integers a and b which are such that $a^b b^a$ is divisible by 2000. What is the least possible value of the product ab?

Solution 4. Let a, b denote positive integers such that $a^a b^b$ is divisible by 2000 and the product ab is minimum. Note that $2, 5$ divide ab, which shows that 10 divides ab, and hence $ab \geq 10$. Using that 2000 divides $10^{10}1^1$, we obtain $ab = 10$.

Now let a, b denote positive integers such that $a^b b^a$ is divisible by 2000 and the product ab is minimum. Note that $2, 5$ divide ab, and hence 10 divides ab. Moreover, $ab \leq 20$ since 2000 divides $4^5 5^4$. If $ab = 10$, then (a, b) is equal to one of

$$(1, 10), (2, 5), (5, 2), (10, 1),$$

and in each of this cases, 2000 does not divide $a^b b^a$. This shows that $ab \neq 10$, and $ab \leq 20$, and ab is a multiple of 10. Observing that 2000 divides $4^5 5^4$, we conclude that $ab = 20$. ■

Exercise 5 (RMO 2000 P7,). Find all real values of a for which the equation $x^4 - 2ax^2 + x + a^2 - a = 0$ has all its roots real.

Solution 5. Note that

$$\begin{aligned}
&x^4 - 2ax^2 + x + a^2 - a \\
&= a^2 - (1 + 2x^2)a + x + x^4 \\
&= (a - x^2 + x - 1)(a - x^2 - x) \\
&= (x^2 - x + 1 - a)(x^2 + x - a) \\
&= \left(\left(x - \frac{1}{2}\right)^2 - \left(a - \frac{3}{4}\right)\right)\left(\left(x + \frac{1}{2}\right)^2 - \left(a + \frac{1}{4}\right)\right),
\end{aligned}$$

which has four real roots if and only if $a - \frac{3}{4}, a + \frac{1}{4}$ are both nonnegative, that is, when $a \geq \frac{3}{4}$. ■