RMO 1999

Exercise 1 (RMO 1999 P2,). Find the number of positive integers which divide 10^{999} but not 10^{998}.

Solution 1. Since every divisor of 10^{998} divides 10^{999}, the required number is equal to the difference of the number of divisors of 10^{999} and 10^{998}, which is equal to $(999+1)(999+1)-(998+1)(998+1)=1999$. ■

Exercise 2 (RMO 1999 P4,). If p, q, r are the roots of the cubic equation $x^3-3px^2+3q^2x-r^3=0$, show that $p=q=r$.

Solution 2. The given conditions imply

$$p+q+r=3p, pq+qr+rp=3q^2, pqr=r^3,$$

which gives

$$q+r=2p, (q+r)^2+2qr=6q^2, (q+r)qr=2r^3.$$

Thus

$$(q-r)(5q+r)=r(q+2r)(q-r)=0$$

holds. If $q\neq r$, then we get

$$5q+r=0, r(q+2r)=0,$$

which gives $q=r=0$. So q, r are equal and hence they are equal to p. ■

Exercise 3 (RMO 1999 P5,). If a, b, c are the sides of a triangle, prove the following inequality:

$$\frac{a}{c+a-b}+\frac{b}{a+b-c}+\frac{c}{b+c-a}\geq 3.$$

Solution 3. Let us use Ravi substitution, that is, let x, y, z be real numbers satisfying

$$a = y + z, b = z + x, c = x + y.$$

Note that x, y, z are positive by the triangle inequality. Observe that

$$\begin{aligned}
&\frac{a}{c+a-b} + \frac{b}{a+b-c} + \frac{c}{b+c-a} \\
&= \frac{y+z}{2y} + \frac{z+x}{2z} + \frac{x+y}{2x} \\
&= \frac{3}{2} + \frac{1}{2}\left(\frac{y}{x} + \frac{z}{y} + \frac{x}{z}\right) \\
&\geq \frac{3}{2} + \frac{3}{2} \quad \text{(by the AM-GM inequality)} \\
&= 3.
\end{aligned}$$

■

Exercise 4 (RMO 1999 P6,). Find all solutions in integers m, n of the equation

$$(m-n)^2 = \frac{4mn}{m+n-1}.$$

Solution 4. Let m, n be integers satisfying $m + n \neq 1$ and the above equation. Note that the above equation is equivalent to

$$(m-n)^2 = \frac{(m+n)^2}{m+n} = m + n,$$

which holds if and only if

$$\frac{1}{2}(m-n)(m-n-1) = n.$$

Writing $m - n = k$, it follows that (m, n) is equal to

$$\left(k + \frac{1}{2}k(k-1), \frac{1}{2}k(k-1)\right) = \left(\frac{1}{2}k(k+1), \frac{1}{2}k(k-1)\right).$$

Also note that for any integer k, the pair

$$\left(\frac{1}{2}k(k+1), \frac{1}{2}k(k-1)\right)$$

is a solution to the given equation if $k^2 \neq 1$. This shows that the solution is

$$\left\{\left(\frac{1}{2}k(k+1), \frac{1}{2}k(k-1)\right) \mid k \in \mathbb{Z} \setminus \{\pm 1\}\right\}.$$

■

The content posted here and at this blog by Evan Chen are quite useful.

Exercise 5 (RMO 1999 P7,). Find the number of quadratic polynomials, $ax^2 + bx + c$, which satisfy the following conditions:

1. a, b, c are distinct,
2. $a, b, c \in \{1, 2, 3, \ldots, 1999\}$ and
3. $x + 1$ divides $ax^2 + bx + c$.

Solution 5. Note the third condition is equivalent to $a + c = b$. So the given problem is equivalent to finding the number of triples (a, b, c) where a, b, c are distinct integers lying between 1 and 1999, and satisfy $a + c = b$. Note that it is equal to the number of pairs (a, c) of distinct integers a, c lying between 1 and 1999, such that their sum $a + c$ is at most 1999. Observe that these are precisely the pairs of the form (a, c) where a, c are integers satisfying $1 \leq a \leq 1998$, $1 \leq c \leq 1999 - a$ and $a \neq c$. Note that the number of such pairs is equal to

$$\sum_{i=1}^{1998} (1999 - i) - 999,$$

where the sum accounts for the pairs (a, c) satisfying $1 \leq a \leq 1998$ and $1 \leq c \leq 1999 - a$, and the term 999 accounts for the pairs (a, c) satisfying these two conditions and the condition $a = c$. We conclude that the number of the quadratic polynomials, satisfying the given conditions, is equal to

$$\begin{aligned}\sum_{i=1}^{1998} (1999 - i) - 999 &= \sum_{i=1}^{1998} i - 999 \\ &= 999 \times 1999 - 999 \\ &= 1998000 - 1998 \\ &= 1996002.\end{aligned}$$

■