RMO 1998

Exercise 1 (RMO 1998 P2,). Let n be a positive integer and $p_1, p_2, \ldots, p_n$ be n prime numbers all larger than 5 such that 6 divides $p_1^2 + p_2^2 + \cdots + p_n^2$. Prove that 6 divides n.

> **Walkthrough** — Observe that any prime larger than 5 is congruent to ± 1 modulo 6.

Solution 1. Note that any prime number larger than 5 is of the form $6k \pm 1$. This yields

$$p_1^2 + p_2^2 + \cdots + p_n^2 \equiv (\pm 1)^2 + (\pm 1)^2 + \cdots + (\pm 1)^2 \bmod 6 \equiv n \bmod 6.$$

Since 6 divides $p_1^2 + p_2^2 + \cdots + p_n^2$, it follows that 6 divides n. ■

Exercise 2 (RMO 1998 P3, , Canadian Mathematical Olympiad 1998 P3). Prove the following inequality for every natural number $n \geq 2$:

$$\frac{1}{n+1}\left(1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n-1}\right) > \frac{1}{n}\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots + \frac{1}{2n}\right).$$

Solution 2. Let us establish the following Claim.

> **Claim** — For any $1 \leq k \leq n$, the inequality
>
> $$\frac{1}{n+1}\left(\frac{1}{2k-1} + \frac{1}{2n-2k+1}\right) \geq \frac{1}{n}\left(\frac{1}{2k} + \frac{1}{2n-2k+2}\right)$$
>
> holds. Moreover, this inequality is strict if $k = 1$ and $n \geq 2$.

Proof of the Claim. Clearing the denominators, it follows that the above inequality is equivalent to

$$4n^2k(n-k+1) \geq (n+1)^2(2k-1)(2n-2k+1).$$

Note that

$$\begin{aligned}
&4n^2k(n-k+1)-(n+1)^2(2k-1)(2n-2k+1)\\
&=n^2(4nk-4k^2+4k-(4nk-4k^2+4k-2n-1))\\
&\quad-(2n+1)(2k-1)(2n-2k+1)\\
&=n^2(2n+1)-(2n+1)(2k-1)(2n-2k+1)\\
&=(2n+1)(n^2-(4nk-4k^2+4k-2n-1))\\
&=(2n+1)((n-2k)^2+2(n-2k)+1)\\
&=(2n+1)(n-2k+1)^2,
\end{aligned}$$

which is nonnegative for any $1 \le k \le n$. This proves the inequality. Moreover, if $k=1$ and $n \ge 2$, then

$$4n^2k(n-k+1)-(n+1)^2(2k-1)(2n-2k+1) > 0,$$

which proves the second statement. □

The Claim implies the given inequality. ■

Exercise 3 (RMO 1998 P5,). Find the minimum possible least common multiple (lcm) of twenty (not necessarily distinct) natural numbers whose sum is 801.

Solution 3. Let L denote the minimum possible least common multiple of twenty (not necessarily distinct) natural numbers, whose sum is 801. Note that one of these natural numbers is larger than 40 (otherwise their sum would be no larger than $40 \times 20 = 800$), and so L is at least 41. Since 801 can be written as

$$3+\underbrace{42+42+\cdots+42}_{19 \text{ times}}=801,$$

we get $L \le 42$. Let us prove the claim below.

Claim — The integer L is not equal to 41.

Proof of the Claim. On the contrary, assume that there exist twenty natural number numbers $x_1, \ldots, x_{20}$ satisfying

$$x_1+\cdots+x_{20}=801, \quad \operatorname{lcm}(x_1,\ldots,x_{20})=41.$$

Since 41 is a prime, it follows that 41 divides x_i for some $1 \le i \le 20$. Since 41 is the least common multiple of $x_1, \ldots, x_{20}$, we obtain $x_i = 41$. Note that the sum $x_1+\cdots+x_{20}$ is less than $20 \cdot 41$. This shows that not all of $x_1+\cdots+x_{20}$ is divisible by 41. Reordering the integers $x_1+\cdots+x_{20}$ (if necessary), let

$1 \leq k < 20$ denote the integer such that none of $x_1, x_2, \ldots, x_k$ is divisible by 41, and $x_{k+1}, x_{k+2}, \ldots, x_{20}$ are equal to 41. We obtain that

$$41 = \operatorname{lcm}(x_1, \ldots, x_{41}) = \operatorname{lcm}(\operatorname{lcm}(x_1, \ldots, x_k), 41).$$

Since none of $x_1, \ldots, x_k$ is divisible by 41, it follows that $\operatorname{lcm}(x_1, \ldots, x_k) = 1$, and hence,

$$x_1 = x_2 = \cdots = x_k = 1.$$

This yields

$$k + 41(20 - k) = 801,$$

which implies that $k \equiv 22 \bmod 41$, which is impossible since $1 \leq k \leq 19$. This proves the Claim. □

We obtain that $L = 42$. ■

Exercise 4 (RMO 1998 P6,  ). Given the 7-element set $A = \{a, b, c, d, e, f, g\}$, find a collection T of 3-element subsets of A such that each pair of elements from A occurs exactly in one of the subsets of T.

Walkthrough — With some trial-and-error, one can find the following example

$$T = \{\{a, b, c\}, \{a, d, e\}, \{a, f, g\}, \{b, d, f\}, \{b, e, g\}, \{c, d, g\}, \{c, e, f\}\},$$

and then check that it has the desired properties.

Suppose we were not able to make a guess for such a set T [a]. In that situation, an approach would be to assume that such a collection T exists, and then try to find out some additional properties of T **using the hypothesis** that T has the stated properties.

- This approach has the **disadvantage** that the argument would rely on the assumption that such a set T exists, which may not be the case [b].
- However, the **advantage** is that we may be able to make further conclusions about such a putative set T, which may in turn allow us to correctly determine such a set T (or even all such sets T), or to even conclude that no such T exists (of course, depending on the problem).

(a) Show that the size of T is equal to 7.

(b) Relabel the elements of A as $1, 2, \ldots, 7$.

(c) Assume that T contains $\{1, 2, 3\}$.

(d) Show that some of the 3-subsets lying in T, other than $\{1, 2, 3\}$, intersects with $\{1, 2, 3\}$ at exactly one element.

(e) Reordering the elements of A if necessary, prove that T contains $\{1, 4, 5\}$.

(f) Next, prove that T contains $\{1,6,7\}$, $\{2,4,6\}$, $\{3,4,7\}$, $\{2,5,7\}$, $\{3,5,6\}$, and conclude that

$$T = \{\{1,2,3\},\{1,4,5\},\{1,6,7\},\{2,4,6\},\{3,4,7\},\{2,5,7\},\{3,5,6\}\}.$$

(g) **Wait!** We have **only** proved that a putative T is equal to the above up to a reordering of the elements of A. This **does not guarantee** the existence of any collection T having the desired properties.

(h) Thus, it still remains to find a collection T with the prescribed properties. Does the above collection of 3-subsets of A work?

[a] Yes, do assume that we were not that clever!
[b] The problem, as stated, indicates the existence of such a T, but this does not suffice.

Solution 4. Suppose T is a set consisting of some size 3-subsets of A such that any size two subset of A is contained in exactly one element of T. Consider the set

$$X = \{(P,Q) \mid P \subseteq Q, |P| = 2, Q \in T\}.$$

Note that X contains precisely $3 \cdot |T|$ many elements. For each subset P of A of size two, there is precisely one element (P,Q) in X. In other words, the map $(P,Q) \mapsto P$ from $X \to \binom{A}{2}$ is a bijection. It follows that $|T| = \frac{1}{3}\binom{7}{2} = 7$, provided there is a set T with the stated properties.

Note that if such a set T exists and it contains $\{a,b,c\}$, then we claim that some of the remaining 3-subsets lying in T has nonempty intersection with $\{a,b,c\}$. Otherwise, the remaining subsets would be subsets of $A \setminus \{a,b,c\}$, and hence T would contain at most $1 + \binom{7-3}{3} = 1 + 4 < 7 = |T|$ elements. This proves the claim. Hence, some of the 3-subsets lying in T, other than $\{a,b,c\}$, intersects with $\{a,b,c\}$ at exactly one element. [1]

Let's relabel the elements of A as $1,2,\ldots,7$ for simplicity. The set T contains $\{1,2,3\}$ and some of the remaining elements of T intersects with $\{1,2,3\}$ at exactly one element. By reordering the elements of A if necessary, we assume that an element of T intersects $\{1,2,3\}$ at $\{1\}$. Note that this element contains none of $2,3$. By reordering the elements of A once again if required, we assume that T contains $\{1,4,5\}$. Since T contains an element containing $\{1,6\}$, it follows that $\{1,6,7\}$ lies in T. Since T contains an element containing $\{2,4\}$, it follows that $\{2,4,6\}$ or $\{2,4,7\}$ lies in T. Reordering $6,7$ if necessary, we assume that T contains $\{2,4,6\}$. Since T contains an element containing $\{3,4\}$, it follows that $\{3,4,i\}$ lies in T for some $1 \leq i \leq 7$ with $i \notin \{1,2,5,6\}$, i.e.,

[1] Having proved this statement, one may realize that it is much easier to establish! Indeed, such a collection T is nonempty, and hence contains the set $\{a,b,c\}$ up to a reordering of the elements of A. By hypothesis, some element of T contains $\{a,d\}$, and that element intersects with $\{a,b,c\}$ at precisely one element. This argument can be used to replace the longer argument above. It should be noted that the above argument introduces a **crucial idea**, namely, to determine the number of certain objects (here the number of elements of T), it is often helpful to take a detour by counting the number of objects of another type (here the number of pairs of the form (P,Q) satisfying suitable conditions). A similar idea is discussed in **??**.

T contains $\{3, 4, 7\}$. Since T contains an element containing $\{2, 5\}$, it follows that $\{2, 5, 7\}$ lies in T. Since T contains an element containing $\{3, 5\}$, it follows that $\{3, 5, 6\}$ lies in T. Using that $|T| = 7$, it follows that

$$T = \{\{1, 2, 3\}, \{1, 4, 5\}, \{1, 6, 7\}, \{2, 4, 6\}, \{3, 4, 7\}, \{2, 5, 7\}, \{3, 5, 6\}\}.$$

Note that the above collection does have the desired properties. ■

Remark. The above argument also shows that up to a reordering of the elements of A, such a set T is equal to

$$\{\{1, 2, 3\}, \{1, 4, 5\}, \{1, 6, 7\}, \{2, 4, 6\}, \{3, 4, 7\}, \{2, 5, 7\}, \{3, 5, 6\}\}.$$