RMO 1997

Exercise 1 (RMO 1997 P2,). For each positive integer n, define $a_n = 20 + n^2$, and $d_n = \gcd(a_n, a_{n+1})$. Find the set of all values that are taken by d_n and show by examples that each of these values are attained.

Solution 1. Let n be a positive integer. Since d_n divides the integers $20 + n^2, 20 + (n+1)^2$, it follows that d_n divides

$$20 + (n+1)^2 - (20 + n^2) = 2n + 1,$$

and d_n also divides

$$20 + (n+1)^2 + 20 + n^2 = 41 + 2n^2 + 2n.$$

This shows that d_n divides

$$2(41 + 2n^2 + 2n) - (2n+1)^2 = 81.$$

So, d_n is equal to one of $1, 3, 9, 27, 81$.

Note that

$$\begin{aligned} d_1 &= \gcd(a_1, a_2) \\ &= \gcd(21, 24) \\ &= 3, \\ d_2 &= \gcd(a_2, a_3) \\ &= \gcd(24, 29) \\ &= 1 \end{aligned}$$

holds.

If $d_n = 9$, then 9 divides $2n + 1$, showing that $n \equiv 4 \bmod 9$. Note that

$$d_4 = \gcd(a_4, a_5) = \gcd(36, 45) = 9$$

holds. If $d_n = 27$, then 27 divides $2n + 1$, implying that $n \equiv 13 \bmod 27$. Note that

$$d_{13} = \gcd(a_{13}, a_{14}) = \gcd(189, 216) = \gcd(189, 216 - 189) = \gcd(189, 27) = 27$$

holds. Also note that if $d_n = 81$, then 81 divides $2n+1$, which shows that $n \equiv 40 \bmod 81$. Observe that

$$\begin{aligned} d_{40} &= \gcd(a_{40}, a_{41}) \\ &= \gcd(20 + 40^2, 20 + 41^2) \\ &= \gcd(1620, 2 \cdot 40 + 1) \\ &= 81 \end{aligned}$$

holds.

This proves that the set of values taken by d_n are precisely

$$1, 3, 9, 27, 81.$$

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Exercise 2 (RMO 1997 P3,). Solve for real x:

$$\frac{1}{[x]} + \frac{1}{[2x]} = x - [x] + \frac{1}{3},$$

where $[x]$ is the greatest integer less than or equal to x (eg, $[3.4] = 3$ and $[-0.8] = -1$).

Solution 2. Let x be a real number with $[x] \neq 0$ satisfying

$$\frac{1}{[x]} + \frac{1}{[2x]} = x - [x] + \frac{1}{3}.$$

Let n denote the integer $[x]$.

If $n \leq x < n + \frac{1}{2}$, then

$$\frac{1}{n} + \frac{1}{2n} = x - n + \frac{1}{3},$$

or equivalently, $x - n = \frac{3}{2n} - \frac{1}{3}$. Using the inequality $0 \leq x - n < \frac{1}{2}$, it follows that $\frac{9}{5} \leq n \leq \frac{9}{2}$, and hence x is equal to one of $2, 3, 4$. If $x = 2$, then

$$x = 2 + \frac{1}{2} + \frac{1}{4} - \frac{1}{3} = 2 + \frac{5}{12} = \frac{29}{12}.$$

If $x = 3$, then

$$x = 3 + \frac{1}{3} + \frac{1}{6} - \frac{1}{3} = \frac{19}{6}.$$

If $x = 4$, then

$$x = 4 + \frac{1}{4} + \frac{1}{8} - \frac{1}{3} = 4 + \frac{1}{24}.$$

If $n+\frac{1}{2} \leq x < n+1$, then we obtain

$$\frac{1}{n} + \frac{1}{2n+1} = x - n + \frac{1}{3},$$

which gives

$$x - n = \frac{1}{n} + \frac{1}{2n+1} - \frac{1}{3}.$$

Using $x \geq n + \frac{1}{2}$, it follows that

$$\frac{1}{n} + \frac{1}{2n+1} - \frac{1}{3} \geq \frac{1}{2},$$

or equivalently, $10n^2 - 7n - 6 < 0$, which yields $-\frac{1}{2} < n < \frac{6}{5}$. Since $n \neq 0$, we get $n = 1$, which implies that

$$x = n + \frac{1}{n} + \frac{1}{2n+1} - \frac{1}{3} = 2.$$

We conclude that x is equal to one of

$$\frac{29}{12}, \frac{19}{6}, \frac{97}{24}, 2.$$

Note that $\frac{29}{12}, \frac{19}{6}, \frac{97}{24}$ are solutions of the given equation. Also note that $x = 2$ does not satisfy the given equation. It follows that the required solutions are

$$\frac{29}{12}, \frac{19}{6}, \frac{97}{24}.$$

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Exercise 3 (RMO 1997 P4,). In a quadrilateral $ABCD$, it is given that AB is parallel to CD and the diagonals AC and BD are perpendicular to each other. Show that

(a) $AD \cdot BC \geq AB \cdot CD$,

(b) $AD + BC \geq AB + CD$.

Solution 3. Note that the inequality $AD \cdot BC \geq AB \cdot CD$ is equivalent to

$$(x^2 + w^2)(y^2 + z^2) \geq (x^2 + y^2)(z^2 + w^2),$$

which reduces to $(x^2 - z^2)(y^2 - w^2) \geq 0$. It suffices to show that $(x-z)(y-w) \geq 0$ holds. Since AB is parallel to CD, it follows that the triangles ABP and CDP are similar, and the lengths of their sides satisfy

$$\frac{x}{z} = \frac{y}{w}.$$

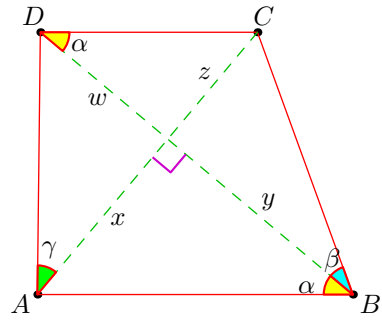


Figure 1: India RMO 1997 P4, Exercise 3

Consequently, the inequality $(x-z)(y-w) \geq 0$ holds.

To prove the second inequality, note that

$$\begin{aligned}(AD+BC)^2 &= AD^2+BC^2+2AD\cdot BC\\ &= x^2+w^2+y^2+z^2+2AD\cdot BC\\ &\geq x^2+y^2+z^2+w^2+2AB\cdot CD\\ &= AB^2+CD^2+2AB\cdot CD\\ &= (AB+CD)^2,\end{aligned}$$

which implies $AD+BC \geq AB+CD$. ■

Exercise 4 (RMO 1997 P5,). Let x, y and z be three distinct real positive numbers. Determine with proof whether or not the three real numbers

$$\left|\frac{x}{y}-\frac{y}{x}\right|, \left|\frac{y}{z}-\frac{z}{y}\right|, \left|\frac{z}{x}-\frac{x}{z}\right|$$

can be the lengths of the sides of a triangle.

Solution 4. Without loss of generality, let us assume that $x > y > z$. Note that

$$\begin{aligned}&z(x^2-y^2)+x(y^2-z^2)-y(x^2-z^2)\\ &= zx(x-z)+(x-z)y^2-y(x^2-z^2)\\ &= (x-z)(zx+y^2-y(x+z))\\ &= (x-z)(y-x)(y-z)\\ &< 0.\end{aligned}$$

By the triangle inequality, it follows that the given three numbers cannot be the lengths of the sides of a triangle. ■

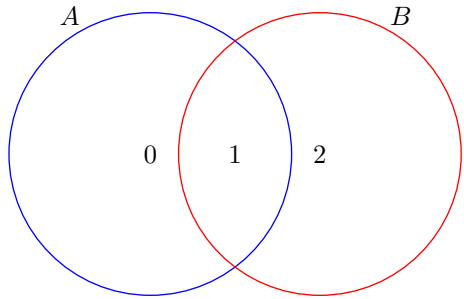


Figure 2: India RMO 1997, Exercise 5

Exercise 5 (RMO 1997 P6,). Find the number of unordered pairs $\{A, B\}$ (that is, the pairs $\{A, B\}$ and $\{B, A\}$ are considered to be the same) of subsets of an n-element set X which satisfy the conditions:

(a) $A \neq B$,

(b) $A \cup B = X$.

(eg., if $X = \{a, b, c, d\}$, then $\{\{a, b\}, \{b, c, d\}\}$, $\{\{a\}, \{b, c, d\}\}$, $\{\emptyset, \{a, b, c, d\}\}$ are some of the admissible pairs.)

> **Walkthrough** — Establish a suitable one-to-one correspondence between the set of the ordered pairs (A, B) with $A \cup B = X$ and the set of maps $f : X \to \{0, 1, 2\}$.

Solution 5. Note that the ordered pairs (A, B) with $A \cup B = X$ are in a one-to-one correspondence with the maps $f : X \to \{0, 1, 2\}$. Such a one-to-one correspondence is given by sending such a pair (A, B) to $f : X \to \{0, 1, 2\}$ taking the values $0, 1, 2$ at the subsets $A \setminus (A \cap B)$, $A \cap B$, $B \setminus (A \cap B)$ of X, respectively (see Fig. 2). This shows that there are 3^n ordered pairs (A, B) satisfying $A \cup B = X$. Among these pairs, there is only one pair satisfying $A = B$, namely, the pair (X, X). Hence, the number of unordered pairs $\{A, B\}$ satisfying $A \neq B$ and $A \cup B = X$ is equal to $\frac{1}{2}(3^n - 1)$. ■