RMO 1996

Exercise 1 (RMO 1996 P2,). Find all triples (a, b, c) of positive integers such that

$$\left(1+\frac{1}{a}\right)\left(1+\frac{1}{b}\right)\left(1+\frac{1}{c}\right)=3.$$

Solution 1. Let a, b, c be positive integers, satisfying the above equation. Since

$$\left(1+\frac{1}{a}\right)\left(1+\frac{1}{b}\right)\left(1+\frac{1}{c}\right)$$

is symmetric with respect to a, b, c, it suffices to consider the case $a \geq b \geq c$. Note that

$$3=\left(1+\frac{1}{a}\right)\left(1+\frac{1}{b}\right)\left(1+\frac{1}{c}\right) \leq \left(1+\frac{1}{c}\right)^3$$

holds. If $c \geq 3$ holds, then we would obtain $3 \leq (1+\frac{1}{c})^3$ implying $81 \leq 64$, which is impossible. This shows that $c=1$ or $c=2$.

Let us consider the case that $c=2$. Then

$$\left(1+\frac{1}{a}\right)\left(1+\frac{1}{b}\right)=2$$

holds, which shows that $2 \leq (1+\frac{1}{b})^2$. This gives $b<3$. Note that $b=1, c=2$ is not possible. This yields $(a, b, c)=(3,2,2)$.

Now, let us consider the case that $c=1$. It follows that

$$\left(1+\frac{1}{a}\right)\left(1+\frac{1}{b}\right)=\frac{3}{2},$$

which implies $\frac{3}{2} \leq (1+\frac{1}{b})^2$, and hence, we get $b<5$. It follows that $b \neq 1$ and $b \neq 2$. Consequently, we obtain that (a, b, c) is equal to one of $(8,3,1),(5,4,1)$.

Note that the triples

$$(3,2,2),(8,3,1),(5,4,1)$$

also satisfy the given equation. It follows that these are precisely all the solutions of the given equation in the positive integers under the hypothesis that $a \geq b \geq c$.

Since $(1+\frac{1}{a})(1+\frac{1}{b})(1+\frac{1}{c})$ is symmetric in a, b, c, the required solutions are obtained by permuting the coordinates of these three solutions, that, the required solutions are precisely

$$(3,2,2),(2,3,2),(2,2,3),$$

$$(8,3,1),(8,1,3),(3,8,1),(3,1,8),(1,8,3),(1,3,8),$$

$$(5,4,1),(5,1,4),(4,5,1),(4,1,5),(1,4,5),(1,5,4).$$

■

Exercise 2 (RMO 1996 P3,). Solve for real number x and y:

$$xy^2 = 15x^2 + 17xy + 15y^2, \quad x^2y = 20x^2 + 3y^2.$$

Solution 2. Adding the first equation multiplied by i to the second, it follows that the given system of equations over the reals is equivalent to

$$x^2y + ixy^2 = (20+15i)x^2 + 17ixy + (3+15i)y^2,$$

which is equivalent to

$$xy(x+iy) = (x+iy)((20+15i)x - i(3+15i)y),$$

which simplies to

$$(x+iy)(20x+15y-xy+3i(5x-y)) = 0.$$

Note that any solution of the above equal is equal to one of $(0,0),(19,95)$, and conversely, any of $(0,0),(19,95)$ satisfies the above equation. Hence, the solutions of the given system of equations over the reals are precisely $(0,0),(19,95)$. ■

Exercise 3 (RMO 1996 P4,). Suppose N is an n-digit positive integer such that

1. all the n-digits are distinct, and
2. the sum of any three consecutive digits is divisible by 5.

Prove that n is at most 6. Further, show that starting with any digit one can find a six-digit number with these properties.

Solution 3. If $n \geq 7$, then note that the first digit of N is congruent to its fourth digit modulo 5, and its fourth digit is congruent to its seventh digit modulo 5, and hence, its first, fourth and seventh digits are congruent to each other modulo 5, which is impossible since the digits of n are distinct and no

three integers among $0, 1, 2, 3, 4, 5, 6, 7, 8, 9$ are congruent modulo 5. This shows that n is at most 6.

Let $0 \leq a \leq 4$ be an integer. Considering the 3-digit numbers

$$140, 230, 320, 410,$$

it follows that there are integers $0 \leq b, c \leq 4$ such that a, b, c are distinct, and their sum is divisible by 5. Taking

$$a_1 = a, a_2 = b, a_3 = c, a_4 = a_1 + 5, a_5 = a_2 + 5, a_6 = a_3 + 5,$$

it follows that the 6-digit integer

$$10^5 a_1 + 10^4 a_2 + 10^3 a_3 + 10^2 a_4 + 10 a_5 + a_6$$

has the required properties. ■

Exercise 4 (RMO 1996 P5,). Let ABC be a triangle and h_a the altitude through A. Prove that

$$(b + c)^2 \geq a^2 + 4h_a^2.$$

Solution 4. Since $\Delta = \frac{1}{2} a h_a$, the inequality reduces to

$$(b + c)^2 - a^2 \geq \frac{1}{a^2} 16\Delta^2,$$

which is equivalent to

$$a^2 \geq (c + a - b)(a + b - c).$$

This follows from the AM-GM inequality. ■

> **Remark.** To prove the inequality $a^2 \geq (c+a-b)(a+b-c)$, one could substitute
>
> $$a = y + z, b = z + x, c = x + y,$$
>
> and note that the above inequality is equivalent to
>
> $$(y + z)^2 \geq 4yz,$$
>
> which holds since $(y - z)^2 \geq 0$.

Exercise 5 (RMO 1996 P6,). Given any positive integer n, show that there are two positive rational numbers a and b, $a \neq b$, which are not integers and which are such that $a - b, a^2 - b^2, a^3 - b^3, \ldots, a^n - b^n$ are all integers.

Walkthrough —

(a) We may expect that $a = c + \frac{1}{2}$ and $b = d + \frac{1}{2}$ would work where c, d are suitable integers. We need to see for what choices of c, d, the difference $a^m - b^m$ is an integer for all $1 \le m \le n$.

(b) By the binomial theorem [a],

$$a^m = c^m + \binom{m}{1} c^{m-1} \frac{1}{2} + \binom{m}{2} c^{m-2} \frac{1}{2^2} + \cdots + \binom{m}{m-1} c \frac{1}{2^{m-1}} + \frac{1}{2^m},$$

$$b^m = d^m + \binom{m}{1} d^{m-1} \frac{1}{2} + \binom{m}{2} d^{m-2} \frac{1}{2^2} + \cdots + \binom{m}{m-1} d \frac{1}{2^{m-1}} + \frac{1}{2^m}.$$

This shows

$$a^m - b^m = (c^m - d^m) + \binom{m}{1} \frac{c^{m-1} - d^{m-1}}{2} + \binom{m}{2} \frac{c^{m-2} - d^{m-2}}{2^2} + \cdots + \binom{m}{m-1} \frac{c-d}{2^{m-1}}.$$

(c) It follows that $a^m - b^m$ would be an integer if we could ensure that

$$\frac{c^{m-1} - d^{m-1}}{2}, \frac{c^{m-2} - d^{m-2}}{2^2}, \ldots, \frac{c-d}{2^{m-1}}$$

are all integers.

(d) Note that $c^{m-1} - d^{m-1}, c^{m-2} - d^{m-2}, \ldots, c - d$ are all divisible by $c - d$. So it would be enough to find suitable integers c, d such that $c - d$ is divisible by $2, 2^2, \ldots, 2^{m-1}$ for all $1 \le m \le n$.

(e) For instance, if we take $c = 2^{n-1}, d = 0$, that is,

$$a = 2^{n-1} + \frac{1}{2}, b = \frac{1}{2},$$

then $a - b, a^2 - b^2, a^3 - b^3, \ldots, a^n - b^n$ are all integers and a, b are positive rationals and not integers.

[a] The binomial theorem says that

$$(x+y)^n = x^n + \binom{n}{1} x^{n-1} y + \binom{n}{2} x^{n-2} y^2 + \cdots + \binom{n}{n-2} x^2 y^{n-2} + \binom{n}{n-1} x y^{n-1} + y^n,$$

where $\binom{n}{i}$ is a positive integer and is equal to $\frac{n!}{i!(n-i)!}$ for $1 \le i \le n-1$. Indeed, if we expand $(x+y)^n$, then it becomes immediately clear that there exist integers $c_1, \ldots, c_{n-1}$ such that

$$(x+y)^n = x^n + c_1 x^{n-1} y + c_2 x^{n-2} y^2 + \cdots + c_{n-2} x^2 y^{n-2} + c_{n-1} x y^{n-1} + y^n$$

holds, and moreover, the integers $c_1, \ldots, c_{n-1}$ do not depend on x and y, they depend only on n. A careful inspection of the above equality shows that for any $1 \le i \le n$, the integer c_i is equal to the number of ways of selecting a set of i

elements from a set of n elements, which is denoted by $\binom{n}{i}$ by convention. This essentially proves (combined with the details to be filled in) the binomial theorem, which states that for any positive integer n,

$$(x+y)^n = x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n-2}x^2y^{n-2} + \binom{n}{n-1}xy^{n-1} + y^n$$

holds. However, it is not immediate that $\binom{n}{i}$ is equal to $\frac{n!}{i!(n-i)!}$. But we do not require it in this problem.

Solution 5. Take $a = 2^m + \frac{1}{2}$ and $b = \frac{1}{2}$, where m is a suitable positive integer to be determined later. Using the binomial theorem, we obtain

$$a^k = 2^{mk} + \binom{k}{1}\frac{2^{m(k-1)}}{2} + \binom{k}{2}\frac{2^{m(k-2)}}{2^2} + \cdots + \binom{k}{k-1}\frac{2^m}{2^{k-1}} + \frac{1}{2^k}$$

for any integer $k \geq 1$. We would like to have that $a-b, a^2-b^2, a^3-b^3, \ldots, a^n-b^n$ are all integers. Note that it suffices to make sure that $m \geq k-1$ for $k = 1, 2, \ldots, n$. Let us take

$$a = 2^n + \frac{1}{2}, b = \frac{1}{2}.$$

Note that the positive rational numbers a, b are not integers, and $a-b, a^2-b^2, a^3-b^3, \ldots, a^n-b^n$ are all integers. ■

Exercise 6 (RMO 1996 P7,). If A is a fifty-element subset of the set $\{1, 2, 3, \ldots, 100\}$ such that no two numbers from A add up to 100, show that A contains a square.

Walkthrough — Decompose $\{1, 2, \ldots, 100\}$ as the union of the subsets $\{1, 99\}$, $\{2, 98\}$, $\ldots$, $\{49, 51\}$, $\{50\}$, $\{100\}$. Does one of these subsets contain two perfect squares?

Solution 6. Note that the set A is formed by choosing at most one element from each of the following sets $\{1, 99\}, \{2, 98\}, \ldots, \{49, 51\}, \{50\}, \{100\}$. If 100 lies in A, then we are done. Otherwise, A is formed by choosing at most one element from the first 50 sets. Since A has 50 elements, one element of each of these 50 sets lies in A. In particular, one of $36, 64$ lies in A, and hence A contains a perfect square. ■