RMO 1995

Exercise 1 (RMO 1995 P2,). Call a positive integer n *good* if there are n integers, positive or negative, and not necessarily distinct, such that their sum and products are both equal to n. Show that the integers of the form $4k+1$ and 4ℓ are good.

Solution 1. Note that 1 is good. Let k be a positive integer. Put

$$a_1 = \cdots = a_k = 1, \quad b_1 = \cdots = b_k = -1.$$

Noting that

$$4k+1 = (4k+1) + 2(a_1 + \cdots + a_k) + 2(b_1 + \cdots + b_k),$$
$$4k+1 = (4k+1)\left(\prod_{i=1}^{k} a_i b_i\right)^2,$$

it follows that $4k+1$ is good. Also note that for a positive integer ℓ, 4ℓ is equal to the sum, and also equal to the product, of $2, 2, \ell$ and $4\ell - 2 - 2 - \ell$ many 1's, and hence, 4ℓ is good. ■

Exercise 2 (RMO 1995 P3, , cf. Tournament of Towns). [**Tao06**, Problem 2.1] Prove that among any 18 consecutive three digit numbers there is at least one number which is divisible by the sum of its digits.

> **Walkthrough** —
>
> **(a)** Show that one among any such consecutive integers is divisible by 18.
>
> **(b)** Prove that its sum of digits, is a multiple of 9, and conclude that it is equal to one of $9, 18, 27$.
>
> **(c)** Show that the sum of its digits is not 27.

Solution 2. Note that among 18 consecutive three digit numbers, there is an integer divisible by 18. Denote it by $n = 100a + 10b + c$ with a, b, c denoting

integers lying between 0 and 9. It follows that 9 divides n, and hence 9 divides $a+b+c$. This shows that $a+b+c$ is equal to one of $9, 18, 27$. Note that $a+b+c = 27$ holds only if $n = 999$. Since 18 divides n, it follows that $a+b+c \neq 27$, and hence, $a+b+c$ is equal to one of $9, 18$. This proves that $a+b+c$ divides n. ■

Exercise 3 (RMO 1995 P4,). Show that the quadratic equation $x^2 + 7x - 14(q^2+1) = 0$, where q is an integer, has no integer root.

Solution 3. It suffices to show that the discriminant of the polynomial $x^2 + 7x - 14(q^2+1)$ is not a perfect square, that is, the integer $49 + 56(q^2+1)$ is not perfect square.

Let us assume that $49 + 56(q^2+1)$ is perfect square. Note that it is divisible by 7. So it is divisible by 49. Hence, q^2+1 is divisible by 7, which is impossible since $7 \equiv 3 \bmod 4$. Alternatively, note that

$$q^2 + 1 \equiv 1, 2, 5, 3 \bmod 7$$

if $q \equiv 0, \pm 1, \pm 2, \pm 3 \bmod 7$, and thus q^2+1 is not divisible by 7. This shows that the discriminant of $x^2 + 7x - 14(q^2+1)$ is not a perfect square. So this polynomial has no integer root. ■

Exercise 4 (RMO 1995 P5,). Show that for any triangle ABC, the following inequality is true:

$$a^2 + b^2 + c^2 > \sqrt{3} \max\{|a^2 - b^2|, |b^2 - c^2|, |c^2 - a^2|\}.$$

Solution 4. Since the above inequality is symmetric with respect to a, b, c, without loss of generality, we assume that $a > b > c$. Thus we need to show that

$$a^2 + b^2 + c^2 > \sqrt{3}(a^2 - c^2)$$

holds. Using the triangle inequality, we obtain $b > a - c > 0$. This yields

$$\begin{aligned} a^2 + b^2 + c^2 - \sqrt{3}(a^2 - c^2) &> a^2 + (a-c)^2 + c^2 - \sqrt{3}(a^2 - c^2) \\ &= 2(a^2 + c^2 - ca) - \sqrt{3}(a^2 - c^2) \\ &= (2 - \sqrt{3})a^2 + (2 + \sqrt{3})c^2 - 2ca \\ &= (2 - \sqrt{3})(a^2 - 2ac(2 + \sqrt{3}) + (2 + \sqrt{3})^2) \\ &= (2 - \sqrt{3})(a - (2 + \sqrt{3})c)^2 \\ &\geq 0. \end{aligned}$$

This completes the proof. ■

The content posted here and at this blog by Evan Chen are quite useful.

Exercise 5 (RMO 1995 P6,). Suppose $A_1A_2\cdots A_{21}$ is a 21-sided regular polygon inscribed in a circle with centre O. How many triangles $A_iA_jA_k$, $1 \le i < j < k \le 21$, contain the centre point O in their interior?

Solution 5. We first count the number of triangles with A_1 as one of the vertices. Suppose $A_1A_iA_j$ is a triangle with O in its interior. Note that both of i, j cannot be > 11. Renaming the vertices if necessary, assume that $2 \le i \le 11$. Let Γ denote the circumcircle of the polygon, and let A' (resp. A_i') denote the diametrically opposite point of A (resp. A_i'). Then A_j has to lie in the small arc $A'A_i'$. Note that there are $(i-1)$ vertices of the 21-sided regular polygon that lie within the small arc $A'A_i'$. So there are

$$(2-1)+(3-1)+\cdots+(11-1) = 1+2+\cdots+10 = 55$$

triangles containing O in their interior and passing through A_1. Hence the total number of triangles satisfying the required condition is equal to $\frac{1}{3}\times 55\times 21 = 385$. ■

Exercise 6 (RMO 1995 P7,). Show that for any real number x:

$$x^2\sin x + x\cos x + x^2 + \frac{1}{2} > 0.$$

Solution 6. When $1+\sin x \neq 0$, we have

$$\begin{aligned}
&x^2\sin x + x\cos x + x^2 + \frac{1}{2}\\
&= (1+\sin x)x^2 + x\cos x + \frac{1}{2}\\
&= (1+\sin x)\left(x + \frac{\cos x}{2(1+\sin x)}\right)^2 + \frac{1}{2} - \frac{\cos^2 x}{4(1+\sin x)}\\
&= (1+\sin x)\left(x + \frac{\cos x}{2(1+\sin x)}\right)^2 + \frac{2+2\sin x - \cos^2 x}{4(1+\sin x)}\\
&= (1+\sin x)\left(x + \frac{\cos x}{2(1+\sin x)}\right)^2 + \frac{(1+\sin x)^2}{4} > 0.
\end{aligned}$$

If $\sin x = -1$, then $x^2\sin x + x\cos x + x^2 + \frac{1}{2}$ is equal to $\frac{1}{2}$ and hence it is positive. This completes the proof. ■

References

[**Tao06**] Terence Tao. *Solving mathematical problems. A personal perspective*. Oxford University Press, Oxford, 2006, pp. xii+103. ISBN: 978-0-19-920560-8; 0-19-920560-4 (cited p. 1)