RMO 1994

Exercise 1 (RMO 1994 P1,). A leaf is torn from a paperback novel. The sum of the numbers on the remaining pages is 15000. What are the page numbers on the torn leaf?

Solution 1. Suppose $1, 2, \ldots, n$ denote the page numbers of the novel and $x, x+1$ denote the page numbers of the torn leaf. The given condition implies that

$$\frac{1}{2}n(n+1) = 15000 + 2x + 1. \tag{1}$$

Since $1 \leq x \leq n-1$, we get

$$15000 + 3 \leq \frac{1}{2}n(n+1) \leq 15000 + 2n - 1,$$

which shows that

$$n^2 + n - (30000 + 6) \geq 0, \quad n^2 - 3n - (30000 - 2) \leq 0.$$

Thus

$$\frac{-1 + \sqrt{120000 + 25}}{2} \leq n \leq \frac{3 + \sqrt{120000 + 1}}{2}.$$

Now it can be checked that [1]

$$\frac{3 + \sqrt{120000 + 1}}{2} < 175, \quad \frac{-1 + \sqrt{120000 + 25}}{2} > 172.$$

Consequently, n is equal to 173 or 174. Since the number of pages of a novel is an even number, we conclude that $n = 174$. This shows that

$$\begin{aligned} 2x &= 87 \times 175 - 15001 \\ &= 80 \times 175 + 50 \times 25 - 15026, \end{aligned}$$

[1] Since n is close to $\frac{1}{2}\sqrt{120000} = 100\sqrt{3}$, which is close 173. So we tried to bound n using integers close to 173, whose squares can be easily computed since squaring 175 is easy, at least once it is multiplied by 2.

and hence

$$\begin{aligned} x &= 40 \times 175 + 25 \times 25 - 7513 \\ &= 7000 + 625 - 7513 \\ &= 112. \end{aligned}$$

Consequently, the page numbers on the torn leaf are $112, 113$[2]. ■

Exercise 2 (RMO 1994 P3,). Find all 6-digit natural numbers $a_1a_2a_3a_4a_5a_6$ formed by using the digits $1, 2, 3, 4, 5, 6$, once each such that the number $a_1a_2 \dots a_k$ is divisible by k, for $1 \leq k \leq 6$.

Walkthrough —

(a) Show that a_2, a_4, a_6 are equal to $2, 4, 6$ in some order.

(b) Prove that $a_5 = 5$, and a_1, a_3 are equal to $1, 3$ in some order.

(c) Using that 3 divides $a_1a_2a_3$, determine a_2.

(d) Using the divisibility condition by 4, show that $a_4 = 6$, and conclude that $a_6 = 4$.

Solution 2. Since $a_1a_2, a_1a_2a_3a_4, a_1a_2a_3a_4a_5a_6$ are divisible by 2, it follows that a_2, a_4, a_6 are even, and hence they are equal to $2, 4, 6$ in some order. Using that $a_1a_2a_3a_4a_5$ is divisible by 5, we get that $a_5 = 5$. So a_1, a_3 are equal to $1, 3$ in some order. Using that $a_1a_2a_3$ is a multiple of 3, we obtain

$$a_1 + a_2 + a_3 \equiv 0 \bmod 3,$$

which yields

$$a_2 \equiv 2 \bmod 3,$$

and hence, $a_2 = 2$ holds. Note that $1234, 3214$ are not divisible by 4. This shows that $a_4 = 6$, and hence $a_6 = 4$. It follows that $a_1a_2a_3a_4a_5a_6$ is equal to 321654, or 123654. Note that the integers $321654, 123654$ satisfy the required conditions too. This proves that $321654, 123654$ are precisely all the 6-digit numbers satisfying the given condition. ■

Exercise 3 (RMO 1994 P4, , All-Russian MO 1991 Grade 10 Day 2 P7). Solve the system of equations for real x and y:

$$5x\left(1 + \frac{1}{x^2 + y^2}\right) = 12, \quad 5y\left(1 - \frac{1}{x^2 + y^2}\right) = 4.$$

[2] Do you find something wrong with it? One may note that substituting $n = 173$ in Equation (1) would yield $x = 25$, implying that the page numbers on the torn leaf are $25, 26$.

The content posted here and at this blog by Evan Chen are quite useful.

Solution 3. Let x, y be real numbers with $(x, y) \neq 0$, and satisfying the given equations. To make use of the factorization $x^2 + y^2 = (x + iy)(x - iy)$, we add the second equation multiplied by i to the first to obtain

$$5\left(x + iy + \frac{1}{x + iy}\right) = 12 + 4i.$$

Substituting $z = x + iy$, we obtain

$$5z^2 - (12 + 4i)z + 5 = 0,$$

which gives

$$\begin{aligned} z &= \frac{1}{10}(12 + 4i \pm \sqrt{(12 + 4i)^2 - 100}) \\ &= \frac{1}{10}(12 + 4i \pm 2\sqrt{7 + 24i}) \\ &= \frac{1}{10}(12 + 4i \pm 2(4 + 3i)). \end{aligned}$$

So

$$z = 2 + i, \frac{2}{5} - \frac{i}{5}.$$

Thus we get $(x, y) = (2, 1), (\frac{2}{5}, -\frac{1}{5})$. ■

Exercise 4 (RMO 1994 P5,). Let A be a set of 16 positive integers with the property that the product of any two distinct numbers of A will not exceed 1994. Show that there are two numbers a and b in A which are not relatively prime.

Walkthrough —

(a) Suppose $\mathcal{A}$ is a set of 16 positive integers, and assume that $\mathcal{A}$ does not satisfy the conclusion of the problem, that is, it is false that **some two** *numbers in* $\mathcal{A}$ **are not** *relatively prime.* In other words, $\mathcal{A}$ has the property that **any two** *numbers in* $\mathcal{A}$ **are** *relatively prime.*

(b) Prove that such a set $\mathcal{A}$ would fail to satisfy the given condition, which states that *the product of* **any two** *distinct elements of* $\mathcal{A}$ *is smaller than* 1994. In other words, show that *the product of* **some two** *elements of* $\mathcal{A}$ *is greater than or equal to* 1994.

Remark. Observe that the above argument shows that if a set of 16 positive integers violates the conclusion of the problem, then it does not satisfy the given condition. **Convince yourself** that **doing so does prove** that if a set of 16 positive integers satisfy the given condition, then **there is no way that** it would fail to satisfy the conclusion.

Further, also **convince yourself** that establishing that the failure of the conclusion forces the failure of the given condition [a] is **equivalent to** establishing

that the given condition implies the stated conclusion.

[a]When there are more than one condition, the given conditions are to be **considered together**, and the *failure of the* **totality** *of the given conditions* means the *failure of* **at least one of** *the given conditions.*

Solution 4. Note that if $n \geq 1$, and $a_1, \ldots, a_n$ are distinct and pairwise coprime positive integers such that $a_i \geq 2$ for all i, then one of them admits a prime factor which is at least as large as the n-th prime. Indeed, for each i, if we fix a prime divisor p_i of a_i, then using that $a_1, \ldots, a_n$ are pairwise coprime, it follows that $p_1, \ldots, p_n$ are distinct primes, and hence the largest among them is at least as large as the n-th prime. Consequently, if $n \geq 2$, and $a_1, \ldots, a_n$ are distinct and pairwise coprime positive integers, then at least $(n-1)$ of them are greater than 1, and hence one of them is divisible by a prime at least as large as the $(n-1)$-st prime.

If possible, let us assume that the elements of A are pairwise coprime. Hence, A contains an element x which is divisible by a prime at least as large as the 15th prime. Similarly, the set $A \setminus \{x\}$ contains an element y which is divisible by a prime at least as large as the 14th prime. Since the 14th and 15th primes are $43, 47$ respectively, it follows that $x \geq 47, y \geq 43$, and consequently,

$$xy \geq 47 \cdot 43 = 2021 > 1994,$$

which contradicts the hypothesis. This proves that there are two numbers a and b in A which are not relatively prime. ■

Remark. Note that the above problem is similar to the following problem in spirit.

Example 5. For a set A of consisting of positive integers, let $\ell(A)$ denote the largest integer which can be expressed as the product of two distinct elements of A. What is the smallest element of the set which consists of the integers of the form $\ell(A)$ as A runs over the sets of size 16 and consisting of positive integers?

Bogus Solution. Since $\ell(A)$ is to be minimized as A runs over the sets 16 pairwise coprime integers and $\ell(A)$ denotes the maximum of the products of the pairs of elements of A, it follows that the minimum value of $\ell(A)$ is achieved precisely when the elements of A are as small as possible. This shows that the minimum value of $\ell(A)$ occurs when

$$A = \{1, 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47\}.$$

Hence, the minimum value of $\ell(A)$ is $43 * 47 = 2021$.

Exercise 6. What goes wrong with the above?

Exercise 7 (RMO 1994 P7,). Find the number of all rational numbers m/n such that

1. $0 < m/n < 1$,
2. m and n are relatively prime,
3. $mn = 25!$.

Walkthrough —

(a) Determine the number of pairs of relatively prime positive integers (m, n) satisfying $mn = 25!$.

(b) Show that such pairs (m, n) with $m > n$, and the pairs with $m < n$ are in one-to-one correspondence.

Solution 5. Note that 25! factors as a product of positive integral powers of the 9 primes $2, 3, 5, 7, 11, 13, 17, 19, 23$. So 25! has 2^9 factorizations as a product of two relatively prime positive integers, that is, there are 2^9 pairs of positive integers (m, n) such that m, n are relatively prime and $mn = 25!$. Certainly, we have either $m > n$ or $m < n$. Moreover, interchanging m, n gives a one-to-one correspondence between the factorizations 25! into relatively prime positive integers m, n with $m > n$ and the factorizations 25! into relatively prime positive integers m, n with $m < n$, that is, the map $(m, n) \mapsto (n, m)$ (that is, interchanging m and n) defines a bijection between the sets

$$S = \{(m, n) | m, n \text{ are relatively prime positive integers satisfying } mn = 25! \text{ and } \frac{m}{n} < 1\},$$

$$T = \{(m, n) | m, n \text{ are relatively prime positive integers satisfying } mn = 25! \text{ and } \frac{m}{n} > 1\}.$$

Since $S \cup T$ has 2^9 elements, it follows that S has cardinality $2^8 = 256$. ■

Exercise 8 (RMO 1994 P8, , All-Russian MO 1991 Grade 11 Day 1 P3). If a, b and c are positive real numbers such that $a + b + c = 1$, prove that

$$(1 + a)(1 + b)(1 + c) \geq 8(1 - a)(1 - b)(1 - c).$$

Solution 6. The given inequality is equivalent to

$$(b + c + 2a)(c + a + 2b)(a + b + 2c) \geq 8(b + c)(c + a)(a + b).$$

Put $x = b + c, y = c + a, z = a + b$. Note that the above inequality can be rewritten as

$$(y + z)(z + x)(x + y) \geq 8xyz,$$

which follows from the AM-GM inequality. ■