RMO 1993

Exercise 1 (RMO 1993 P2,). Prove that the ten's digit of any power of 3 is even.

Solution 1. For any positive integer n, note that

$$\begin{aligned} 3^{2n} &= (10-1)^n \\ &= \sum_{i=0}^{n} \binom{n}{i} 10^{n-i}(-1)^i \\ &\equiv \binom{n}{n-1} 10(-1)^{n-1} + (-1)^n \pmod{100} \\ &\equiv 10(-1)^{n-1}n + (-1)^n \pmod{100} \\ &\equiv \begin{cases} (10-n)10+1 \pmod{100} & \text{if } n \text{ is even,} \\ (n-1)10+9 \pmod{100} & \text{if } n \text{ is odd.} \end{cases} \end{aligned}$$

Writing $n = 10k + r$ for some integers k, r with $1 \le r \le 10$, it follows that any power of 3 with a positive even exponent has an even number in its ten's digit.

For any positive integer n, observe that

$$3^{2n+1} \equiv \begin{cases} 3(10-n)10+3 \pmod{100} & \text{if } n \text{ is even,} \\ (3(n-1)+2)10+7 \pmod{100} & \text{if } n \text{ is odd.} \end{cases}$$

Also note that the ten's digit of 3 is even. It follows that any power of 3 with a positive odd exponent has an even number in its ten's digit. ■

Exercise 2 (RMO 1993 P3,). Suppose $A_1A_2\cdots A_{20}$ is a 20-sided regular polygon. How many non-isosceles (scalene) triangles can be formed whose vertices are among the vertices of the polygon but whose sides are not the sides of the polygon?

Walkthrough —

(a) Determine the number of the triangles with vertices among those of $A_1A_2\cdots A_{20}$.

(b) Determine the number of the triangles with vertices among those of $A_1A_2\cdots A_{20}$, which share at most two sides with the polygon.

(c) Determine the number of the triangles with vertices among those of $A_1A_2\cdots A_{20}$, which share no side with the polygon.

(d) Observe that there are no equilateral triangles among them (why?), and determine the number of isosceles triangles among them.

Solution 2. There are $\binom{20}{3}$ triangles whose vertices are among the vertices of the polygon. Among them, there are 20 triangles that share two sides with the polygon and there are $20\times(20-4)$ triangles that share only side with the polygon. So there are $\binom{20}{3}-20-20\times(20-4)$ triangles whose vertices are among the vertices of the polygon, but whose sides are not the sides of the polygon. Among these, there are no equilateral triangles and there are $20\times\left(\frac{1}{2}(20-2)-1\right)=160$ triangles that are isosceles. So there are

$$\binom{20}{3}-20-20\times(20-4)-160=1140-500=640$$

scalene triangles whose vertices are among the vertices of the polygon but whose sides are not the sides of the polygon. ■

Exercise 3 (RMO 1993 P4,). Let $ABCD$ be a rectangle with $AB=a$ and $BC=b$. Suppose r_1 is the radius of the circle passing through A and B and touching CD; and similarly r_2 is the radius of the circle passing through B and C and touching AD. Show that

$$r_1+r_2\geq\frac{5}{8}(a+b).$$

Solution 3. Since QS is perpendicular to AD, it is parallel to CD. Let SQ produced meet BC at R. Since RS is parallel to CD and $\angle RCD$ is a right-angle, it follows that QR is perpendicular to BC. This gives $QR^2+RC^2=QC^2$, which yields

$$(a-r_2)^2+\left(\frac{b}{2}\right)^2=r_2^2.$$

This implies that

$$r_2=\frac{a}{2}+\frac{b^2}{8a}.$$

A similar argument shows that

$$r_1=\frac{b}{2}+\frac{a^2}{8b}.$$

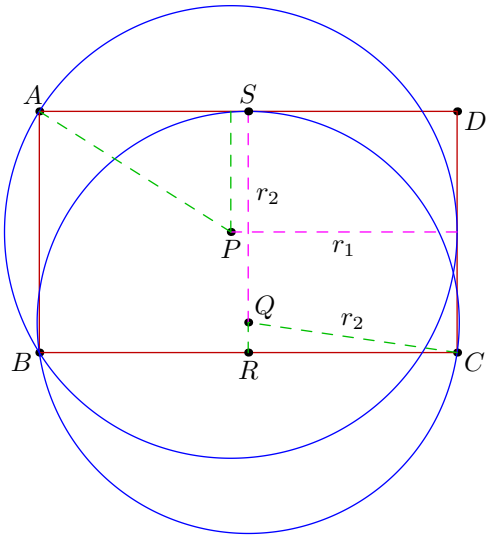


Figure 1: India RMO 1993 P4, Exercise 3

It follows that

$$\begin{aligned} r_1 + r_2 &= \frac{1}{2}(a+b) + \frac{1}{8}\left(\frac{a^2}{b} + \frac{b^2}{a}\right) \\ &\geq \frac{1}{2}(a+b) + \frac{1}{8}\frac{(a+b)^2}{a+b} \quad \text{(by the Cauchy–Schwarz inequality)} \\ &= \frac{5}{8}(a+b). \end{aligned}$$

This completed the proof. ■

Exercise 4 (RMO 1993 P5,). Show that $19^{93} - 13^{99}$ is a positive integer divisible by 162.

Solution 4. Note that the congruences

$$\begin{aligned} 19^{93} &\equiv (1+18)^{93} \bmod 9^2 \\ &\equiv 1 + 93 \cdot 18 \bmod 9^2 \\ &\equiv 1 + 12 \cdot 18 \bmod 9^2 \\ &\equiv 1 + 8 \cdot 27 \bmod 9^2 \\ &\equiv 1 - 27 \bmod 9^2 \\ &\equiv -26 \bmod 9^2 \\ &\equiv 55 \bmod 9^2, \end{aligned}$$

$$\begin{aligned} 13^{99} &\equiv (4+9)^{99} \bmod 9^2 \\ &\equiv 4^{99} + 99 \cdot 4^{98} \cdot 9 \bmod 9^2 \\ &\equiv 4^{99} \bmod 9^2 \end{aligned}$$

hold. Also note that

$$\begin{aligned} 4^{81} &\equiv 1 + 81 \cdot 3 + \binom{81}{2} 3^2 + \binom{81}{3} 3^3 \bmod 9^2 \\ &\equiv 1 \bmod 9^2 \end{aligned}$$

holds, which yields

$$\begin{aligned} 4^{99} &\equiv 4^{81} \cdot 4^{18} \bmod 81 \\ &\equiv 4^{18} \bmod 81 \\ &\equiv (1+3)^{18} \bmod 81 \\ &\equiv 1 + 18 \cdot 3 + \frac{18 \cdot 17}{2} 3^2 + \frac{18 \cdot 17 \cdot 16}{6} 3^3 \bmod 81 \\ &\equiv 1 + 18 \cdot 3 \bmod 81 \\ &\equiv 55 \bmod 81. \end{aligned}$$

It follows that $19^{93} - 4^{99}$ is divisible by 81, and hence, $19^{93} - 13^{99}$ is divisible by 81. Since $19^{93} - 13^{99}$ is an even number, and $2, 81$ are coprime, we conclude that $19^{93} - 13^{99}$ is divisible by 162. ■

Exercise 5 (RMO 1993 P6,). If a, b, c, d are four positive real numbers such that $abcd = 1$, prove that

$$(1+a)(1+b)(1+c)(1+d) \geq 16.$$

Solution 5. It follows by applying the AM-GM inequality to the factors and using $abcd = 1$. ■

Exercise 6 (RMO 1993 P7,). In a group of ten persons, each person is asked to write the sum of the ages of all the other 9 persons. If all the ten sums form the 9-element set $\{82, 83, 84, 85, 87, 89, 90, 91, 92\}$, find the individual ages of the persons (assuming them to be whole numbers of years).

Solution 6. The ten sums are

$$82, 83, 84, 85, 87, 89, 90, 91, 92, n$$

for a unique $n \in \{82, 83, 84, 85, 87, 89, 90, 91, 92\}$. Denote the individual ages by $a_1, \ldots, a_{10}$. Reordering them if necessary, we have

$$(a_1 + \cdots + a_{10}) - a_1 = 82,$$

$$\begin{aligned}(a_1+\cdots+a_{10})-a_2&=83,\\(a_1+\cdots+a_{10})-a_3&=84,\\(a_1+\cdots+a_{10})-a_4&=85,\\(a_1+\cdots+a_{10})-a_5&=87,\\(a_1+\cdots+a_{10})-a_6&=89,\\(a_1+\cdots+a_{10})-a_7&=90,\\(a_1+\cdots+a_{10})-a_8&=91,\\(a_1+\cdots+a_{10})-a_9&=92,\\(a_1+\cdots+a_{10})-a_{10}&=n.\end{aligned}$$

Adding them yields

$$9(a_1+\cdots+a_{10})=82+83+84+85+87+89+90+91+92+n.$$

Since the integer 9 divides $82+83+84+85+87+89+90+91+92+n$, it follows that

$$1+2+3+4+6+8+0+1+2+n$$

is divisible by 9, which shows that n is divisible by 9. Among the integers $82, 83, 84, 85, 87, 89, 90, 91, 92$, only 90 is divisible by 9. It follows that $n=90$. Thus

$$\begin{aligned}a_1+\cdots+a_{10}&=\frac{1}{9}(82+83+84+85+87+89+90+91+92+90)\\&=\frac{1}{9}(10\times 81+1+2+3+4+6+8+9+10+11+9)\\&=\frac{1}{9}(810+63)=97.\end{aligned}$$

This gives

$$a_1=15, a_2=14, a_3=13, a_4=12, a_5=10,$$
$$a_6=8, a_7=7, a_8=6, a_9=5, a_{10}=7.$$

■

Exercise 7 (RMO 1993 P8,). I have 6 friends and during a vacation I met them during several dinners. I found that I dined with all the 6 exactly on 1 day; with every 5 of them on 2 days; with every 4 of them on 3 days; with every 3 of them on 4 days; with every 2 of them on 5 days. Further every friend was present at 7 dinners and every friend was absent at 7 dinners. How many dinners did I have alone?

Walkthrough —

(a) Denote the friends by $F_1, \ldots, F_6$. For $1 \leq i \leq 6$, let D_i denote the set of the dinners where F_i was present.

(b) Show that the total number of dinners is 14.

(c) Let n denote the number of dinners that I had alone.

(d) Note that the size of the set $A_1 \cup \cdots \cup A_6$ is equal to $14 - n$.

Solution 7. Denote the friends by $F_1, \ldots, F_6$. For $1 \leq i \leq 6$, let D_i denote the set of the dinners where F_i was present. Let n denote the number of dinners that I had alone. Since every friend was present in 7 dinners, and was absent in 7 dinners, it follows that the total number of dinners is 14. Consequently, the size of the set $A_1 \cup \cdots \cup A_6$ is equal to $14 - n$. By the inclusion-exclusion principle, we obtain

$$\begin{aligned}
14 - n &= \sum_{i=1}^{6} |A_i| - \sum_{1 \leq i_1 < i_2 \leq 6} |A_{i_1} \cap A_{i_2}| \\
&\quad + \sum_{1 \leq i_1 < i_2 < i_3 \leq 6} |A_{i_1} \cap A_{i_2} \cap A_{i_3}| - \sum_{1 \leq i_1 < i_2 < i_3 < i_4 \leq 6} |A_{i_1} \cap A_{i_2} \cap A_{i_3} \cap A_{i_4}| \\
&\quad + \sum_{1 \leq i_1 < i_2 < i_3 < i_4 < i_5 \leq 6} |A_{i_1} \cap A_{i_2} \cap A_{i_3} \cap A_{i_4} \cap A_{i_5}| \\
&\quad - |A_{i_1} \cap A_{i_2} \cap A_{i_3} \cap A_{i_4} \cap A_{i_5} \cap A_{i_6}| \\
&= \binom{6}{1}7 - \binom{6}{2}5 + \binom{6}{3}4 - \binom{6}{4}3 + \binom{6}{5}2 - 1 \\
&= 42 - 75 + 80 - 45 + 12 - 1 \\
&= 13.
\end{aligned}$$

This shows that $n = 1$. In other words, I dined alone only on one day. ■