RMO 1992

Exercise 1 (RMO 1992 P1,). Determine the set of integers n for which $n^2 + 19n + 92$ is a square of an integer.

Solution 1. Let n be an integer such that $n^2 + 19n + 92 = m^2$ holds for some non-negative integer m. This gives

$$(2n + 19)^2 + 7 = (2m)^2,$$

which yields

$$7 = (2m - 2n - 19)(2m + 2n + 19).$$

Since $2m+2n+19$ is positive, it follows that $(2m-2n-19, 2m+2n+19)$ is equal to $(7, 1)$ or $(1, 7)$. This shows that (m, n) is equal to one of $(2, -11), (2, -8)$, and consequently, n is equal to one of $-11, -8$. Note that

$$(-11)^2 - 19 \cdot 11 + 92 = 4, \quad (-8)^2 - 19 \cdot 8 + 92 = 4$$

holds. This proves that the integers satisfying the given condition are precisely $-11, -8$. ■

Exercise 2 (RMO 1992 P2, , cf. Moscow Mathematical Olympiad 1973 Grade 8 First Round P4). If $\frac{1}{a} + \frac{1}{b} = \frac{1}{c}$, where a, b, c are positive integers with no common factor, prove that $(a + b)$ is the square of an integer.

> **Walkthrough** — Is the given equation equivalent to
>
> $$(a - c)(b - c) = c^2?$$

Solution 2. Let a, b, c be positive integers satisfying the given equation. Assume that a, b have no common prime factors. Note that $(a - c)(b - c) = c^2$ holds. Also note that any common prime divisor of $a - c, b - c$ divides $(a - c)(b - c) = c^2$, and hence it divides the integers a, b, which is impossible. This shows that the integers $a - c, b - c$ are relatively prime, and satisfy $(a - c)(b - c) = c^2$. Note also that $a > c$ holds. Hence, there exist relatively

prime positive integers x, y such that $c = xy$, $a - c = x^2$ and $b - c = y^2$ holds. This gives

$$a = c + x^2 = xy + x^2, \quad b = c + y^2 = xy + y^2.$$

This implies that $a + b$ is a perfect square. ■

Exercise 3 (RMO 1992 P3,). Determine the largest 3-digit prime factor of the integer $\binom{2000}{1000}$.

Solution 3. Suppose there is a 3-digit prime p which divides the integer $\binom{2000}{1000}$. Note that

$$\binom{2000}{1000} = \frac{2000 \cdot 1999 \cdot 1998 \cdot \cdots \cdot 1001}{1000 \cdot 999 \cdot 998 \cdot \cdots \cdot 1}$$

holds. If $p > 500$, then using the fact that p divides $\binom{2000}{1000}$, it follows that $3p \leq 2000$, which implies $p \leq 666$, and hence, $p \leq 661$. This shows that any 3-digit prime divisor of $\binom{2000}{1000}$, larger than 500, is at most 661.

Note that 661 is a 3-digit prime, and it divides 1000!. Since $1000 > 661 > 500$ holds, it follows that $2000 > 2 \cdot 661 \geq 1001$. Also note that $3 \cdot 661 < 2000$. This implies that the highest power of 661 dividing 1000! is 661, and the highest power of 661 dividing $2000 \cdot 1999 \cdot 1998 \cdot \cdots \cdot 1001$ is at least 661^2. Consequently, 661 divides $\binom{2000}{1000}$.

This proves that 661 is the largest 3-digit prime factor of the integer $\binom{2000}{1000}$. ■

Exercise 4 (RMO 1992 P7,). Show that

$$1 < \frac{1}{1001} + \frac{1}{1002} + \cdots + \frac{1}{3001} < \frac{4}{3}$$

Solution 4. To obtain the lower bound, note that

$$\begin{aligned}
&\frac{1}{1001} + \frac{1}{1002} + \cdots + \frac{1}{3001} \\
&> \frac{1}{1001} + \frac{1}{1002} + \cdots + \frac{1}{3000} \\
&= \left(\frac{1}{1001} + \cdots + \frac{1}{1250}\right) + \left(\frac{1}{1251} + \cdots + \frac{1}{1500}\right) + \left(\frac{1}{1501} + \cdots + \frac{1}{1750}\right) \\
&+ \left(\frac{1}{1751} + \cdots + \frac{1}{2000}\right) + \left(\frac{1}{2001} + \cdots + \frac{1}{2250}\right) + \left(\frac{1}{2251} + \cdots + \frac{1}{2500}\right) \\
&+ \left(\frac{1}{2501} + \cdots + \frac{1}{2750}\right) + \left(\frac{1}{2751} + \cdots + \frac{1}{3000}\right) \\
&> \frac{250}{1250} + \frac{250}{1500} + \frac{250}{1750} + \frac{250}{2000} + \frac{250}{2250} + \frac{250}{2500} + \frac{250}{2750} + \frac{250}{3000}
\end{aligned}$$

The content posted here and at this blog by Evan Chen are quite useful.

$$> \frac{1}{5}+\frac{1}{6}+\cdots+\frac{1}{12} = 17\left(\frac{1}{60}+\frac{1}{66}+\frac{1}{70}+\frac{1}{72}\right) > 1.$$

To get the upper bound, note that

$$\begin{aligned}
&\frac{1}{1001}+\frac{1}{1002}+\cdots+\frac{1}{3001}\\
&<\frac{1}{1000}+\frac{1}{1001}+\cdots+\frac{1}{3000}\\
&<\left(\frac{1}{1000}+\frac{1}{1001}+\cdots+\frac{1}{1499}\right)+\left(\frac{1}{1500}+\frac{1}{1501}+\cdots+\frac{1}{1999}\right)\\
&+\left(\frac{1}{2000}+\frac{1}{2001}+\cdots+\frac{1}{2499}\right)+\left(\frac{1}{2500}+\frac{1}{2501}+\cdots+\frac{1}{2999}\right)+\frac{1}{3000}\\
&<\frac{500}{1500}+\frac{500}{2000}+\frac{500}{2500}+\frac{500}{3000}+\frac{1}{3000}\\
&=\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{3000}\\
&=\frac{57}{60}+\frac{1}{3000}\\
&<1\\
&<\frac{4}{3}.
\end{aligned}$$

■

Exercise 5 (RMO 1992 P8,). Solve the system

$$\begin{aligned}
(x+y)(x+y+z) &= 18\\
(y+z)(x+y+z) &= 30\\
(z+x)(x+y+z) &= 2A
\end{aligned}$$

in terms of the parameter A.

Solution 5. The given system can be rewritten as

$$\begin{aligned}
(x+y+z)^2-z(x+y+z) &= 18\\
(x+y+z)^2-x(x+y+z) &= 30\\
(x+y+z)^2-y(x+y+z) &= 2A.
\end{aligned}$$

Adding them, we obtain $(x+y+z)^2 = 24+A$. Note that

$$\begin{aligned}
x(x+y+z) &= (x+y+z)^2-30\\
&= -6+A,\\
y(x+y+z) &= (x+y+z)^2-2A
\end{aligned}$$

$$\begin{aligned} &= 24 - A, \\ z(x+y+z) &= (x+y+z)^2 - 18 \\ &= 6 + A \end{aligned}$$

holds. Note that $x + y + z \neq 0$, which gives $A \neq -24$ and hence, we obtain

$$(x, y, z) = \left(\pm \frac{-6 + A}{\sqrt{24 + A}}, \pm \frac{24 - A}{\sqrt{24 + A}}, \pm \frac{6 + A}{\sqrt{24 + A}} \right)$$

where the signs $+$ and $-$ correspond, and $\sqrt{24 + A}$ denotes a square root of $24 + A$ in $\mathbb{C}$. ■