RMO 1991

Exercise 1 (RMO 1991 P2,). If a, b, c and d are any 4 positive real numbers, then prove that

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{a} \geq 4.$$

Solution 1. Note that

$$\begin{aligned}\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{a} &\geq 2\sqrt{\frac{a}{c}}+2\sqrt{\frac{c}{a}} \\ &\geq 4.\end{aligned}$$

■

Exercise 2 (RMO 1991 P3,). A four-digit number has the following properties:

1. it is a perfect square,
2. its first two digits are equal to each other,
3. its last two digits are equal to each other.

Find all such four-digit numbers.

Solution 2. Let n be a positive integer satisfying the given conditions. Denote the first two digits of n by a, and the last two digits of n by b. Since n is a perfect square, it follows that n is congruent to one of $0, 1$ modulo 4, which shows that $10b + b = 11b$ is congruent to one of $0, 1$ modulo 4, or equivalently, b is congruent to one of $0, 3$ modulo 4. Similarly, using that a perfect square is congruent to one of $0, 1, -1$ modulo 5, it follows that b is congruent to one of $-1, 0, 1$ modulo 5. This implies that b is equal to one of $0, 4$.

If $b = 0$, then n is the square of a multiple of 10, and hence, its first two digits would not be equal. It follows that $b = 4$, and consequently, we obtain

$$n = 1000a + 100a + 10b + b = 1100a + 11b = 1100a + 44.$$

This shows that 11 divides the perfect square n. So, 11^2 divides n, and hence, 11 divides $100a + 4$, which yields

$$a \equiv 7 \bmod 11.$$

Since $1 \leq a \leq 9$, we obtain $a = 7$, and this gives $n = 7744$. Note that $7744 = 88^2$, it follows that 7744 is the only four-digit number satisfying the given conditions. ■

Exercise 3 (RMO 1991 P4,). There are two urns each containing an arbitrary number of balls. Both are non-empty to begin with. We are allowed two types of operations:

1. remove an equal number of balls simultaneously from the urns, and
2. double the number of balls in any one of them.

Show that after performing these operations finitely many times, both the urns can be made empty.

Walkthrough —

(a) If both the urns have an equal number of balls, then removing all the balls from both the urns, we are done.

(b) Otherwise, remove enough balls from both the urns such that one of them is left with only one ball, and the other has one or more balls.

(c) Double the number of balls in the urn containing only one ball, and then go back to the first step.

Solution 3. Denote the urns by A, B and denote the number of balls in them by $n(A), n(B)$ respectively. Without loss of generality, assume that $n(A) \geq n(B)$. Note that it suffices to consider the case $n(B) = 1$ since if we take out $n(B) - 1$ many balls from both the urns (note that $n(B) \geq 1$ and thus $n(B) - 1 \geq 0$), then $n(A) - n(B) + 1$ many balls are left in urn A and exactly one ball is left in urn B.

We apply induction on $n(A)$ to complete the proof. If $n(A) = 1$, then applying the first operation, both the urns can be made empty. Assume that $k \geq 1$ is a positive integer such that both the urns can be made empty by making a finite number of operations of the stated type whenever one urn contains only one ball, and the other urn contains at most k balls.

Suppose there are two urns, denoted by A and B, and the number of balls in them, denoted $n(A), n(B)$ respectively, satisfy $n(A) = k + 1, n(B) = 1$. Now let us keep on doubling the number of balls in urn B until the number of balls in it is greater than or equal to the number of balls in urn A, that is, we double the number of balls in urn B for r times where r is the least positive integer such that $2^r \geq k + 1$. Let us remove k balls from both the urns, and then only

one ball is left in urn A and $2^r - k$ balls are left in urn B. Note that $2^r - k < k$ since r is the least positive integer such that $2^r \geq k + 1$. Thus we are left with two urns, one of them containing one ball, and the other containing at most k balls. By the induction hypothesis, these urns can be made empty performing a finite number of operations of the stated type. ■

Exercise 4 (RMO 1991 P6,). Find all integer values of a such that the quadratic expression

$$(x + a)(x + 1991) + 1$$

can be factored as a product $(x + b)(x + c)$ where b and c are integers.

Solution 4. Let us establish the following claim.

> **Claim —** A monic quadratic polynomial $f(x)$ with integer coefficients factorizes into the product of two monic polynomials with integer coefficients if and only if the discriminant of $f(x)$ is a perfect square.

Proof of the Claim. Since the discriminant of $f(x)$ is equal to the square of the twice of the difference of its roots, the "only if part" follows.

To prove the "if part", assume that the discriminant of $f(x)$ is a perfect square. Write $f(x) = x^2 + Bx + C$. Note that $B^2 - 4C = n^2$ for some integer n. This shows that the integers B, n are of the same parity. It follows that

$$-B + n, -B - n$$

are even integers. This implies that the polynomial $f(x)$ have integer roots, proving the "if part" of the Claim. □

By the above Claim, the given problem is equivalent to determining all the integers a such that the discriminant of

$$(x + a)(x + 1991) + 1$$

is a perfect square, that is,

$$(a + 1991)^2 - 4(1991a + 1) = n^2$$

holds for some integer n.

Let a, n be integers satisfying the above condition. Note that the above can be rewritten as

$$(a - 1991)^2 - n^2 = 4,$$

which is equivalent to

$$(a - 1991 - n)(a - 1991 + n) = 4.$$

Noting that the integers $a-1991-n, a-1991+n$ are of the same parity, it follows that the above condition is equivalent to $(a-1991-n, a-1991+n)$ being equal to one of $(2,2), (-2,-2)$, which holds if and only if (a,n) is equal to one of $(1993,0), (1989,0)$.

This proves that the integer values of a satisfying the required condition are precisely $1989, 1993$. ■

Exercise 5 (RMO 1991 P7, , cf. All-Russian MO 1989 Grade 8 Day 2 P5 ??, India BStat 2006 P3). [Eng98, p. 121] Show that n^4+4^n is prime if and only if $n=1$.

Solution 5. Let n be a positive integer such that n^4+4^n is a prime. Note that n is odd, and

$$\begin{aligned} n^4+4^n &= (n^2+2^n)^2 - 2^{n+1}n^2 \\ &= (n^2+2^n)^2 - (2^{(n+1)/2}n)^2 \\ &= (n^2+2^n-2^{(n+1)/2}n)(n^2+2^n+2^{(n+1)/2}n) \end{aligned}$$

holds. Since $n^4+4^n, n^2+2^n+2^{(n+1)/2}n$ are positive, it follows that $n^2+2^n-2^{(n+1)/2}n$ is positive. This gives $n^2+2^n-2^{(n+1)/2}n=1$. Note that

$$n^2+2^n-2^{(n+1)/2}n \geq 2\cdot 2^{n/2}n - 2^{(n+1)/2}n = (\sqrt{2}-1)2^{(n+1)/2}n$$

holds. If $n\geq 2$, then the above yields

$$n^2+2^n-2^{(n+1)/2}n \geq (\sqrt{2}-1)2^{(n+1)/2}n > 1,$$

which implies that n^4+4^n is not a prime. This proves that $n=1$. Moreover, if $n=1$, then n^4+4^n is a prime. This completes the proof. ■

Exercise 6 (RMO 1991 P8,). The 64 squares of an 8×8 chessboard are filled with positive integers in such a way that each integer is the average of the integers on the neighbouring squares. (Two squares are neighbours if they share a common edge or a common vertex. Thus a square can have $8, 5$ or 3 neighbours depending on its position). Show that all the 64 integer entries are in fact equal.

Walkthrough —

(a) Consider a square containing the maximum (or minimum) of all the entries, denoted by m.

(b) Show that the entries of the neighbouring squares are equal to m.

(c) What are the possibilities for the entries of the remaining squares?

Solution 6. Let m denote the maximum among the entries of the 64 squares. Note that if a square contains m, then the entries of its neighbouring squares are equal to m (otherwise, the entry of some of the neighbouring squares would be strictly smaller than m, then the average of the entries of the neighbouring squares would be strictly smaller than m, which is impossible). Consequently, the entries of any two neighbouring squares are equal to m if one of them contains m. Let S denote a square containing m. Note that any other square on the chessboard can be reached from S through a sequence of squares such that the successive squares are neighbours, that is, given any square T other than S, there is a sequence of squares

$$S_1, S_2, \ldots, S_n \quad \text{with } S_1 = S, S_n = T$$

such that S_{i+1} is a neighbour of S_i for all $1 \leq i < n$. Then by the above argument, the entries of the squares $S_1, S_2, \ldots, S_n$ are equal to m. ■

References

[**Eng98**] Arthur Engel. *Problem-solving strategies*. Problem Books in Mathematics. Springer-Verlag, New York, 1998, pp. x+403. ISBN: 0-387-98219-1 (cited p. 4)