RMO 1990

Exercise 1 (RMO 1990 P1,). Two boxes contain between them 65 balls of several different sizes. Each ball is white, black, red or yellow. If you take any 5 balls of the same colour at least two of them will always be of the same size (radius). Prove that there are at least 3 balls which lie in the same box have the same colour and have the same size (radius).

> **Walkthrough** —
>
> **(a)** At least how many balls are there in the box with more balls?
>
> **(b)** At least how many balls are there in a **large** colour class in this box?
>
> **(c)** What about the sizes of the balls of this colour class?

Solution 1. By the pigeonhole principle, one box contains at least 33 balls. Applying the pigeonhole principle once again, it follows that at least 9 balls in this box are of the same colour. Since among five balls of the same colour, at least two of them are of the same size we conclude that these 9 balls of the same colour are of at most four different sizes. By the pigeonhole principle, at least three of these 9 balls are of the same size. ■

Exercise 2 (RMO 1990 P2, , Nesbitt's inequality 1903, Moscow Mathematical Olympiad 1963 Grade 9 Second Round P3, UK BMO 1976 P2). Let $a, b, c > 0$. Then

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}.$$

This has many proofs, for instance using AM-GM, AM-HM, Cauchy-Schwarz inequality, rearrangement inequality. We present a quick proof from [**Hun08**].

Solution 2. Put

$$\alpha = \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b},$$
$$\beta = \frac{b}{b+c} + \frac{c}{c+a} + \frac{a}{a+b},$$

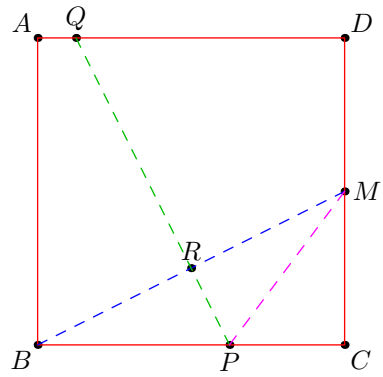


Figure 1: RMO 1990 P3, Exercise 3

$$\gamma = \frac{c}{b+c} + \frac{a}{c+a} + \frac{b}{a+b}.$$

By the AM-GM inequality, the inequalities $\alpha + \beta \geq 3, \beta + \gamma \geq 3, \gamma + \alpha \geq 3$ hold. Adding them together yields $2(\alpha + \beta + \gamma) \geq 9$. Using $\beta + \gamma = 3$, we obtain $\alpha \geq 3/2$. ■

Exercise 3 (RMO 1990 P3,). A square sheet of paper $ABCD$ is so folded that B falls on the mid-point M of CD. Prove that the crease will divide BC in the ratio $5 : 3$.

Solution 3. Let the crease intersect BC at P, and AD at Q. Let R denote the midpoint of BM. Then PQ is the perpendicular bisector of BM. Note that $\triangle BPR$ and $\triangle MPR$ are congruent. This gives $BP = PM$. Applying the Pythagoras' theorem to $\triangle CPM$, we obtain

$$CM^2 + CP^2 = PM^2,$$

which gives

$$CM^2 + CP^2 = BP^2.$$

This implies that

$$CM^2 = (BP - CP)(BP + CP),$$

which yields

$$BP - CP = \frac{CM^2}{BP + CP} = \frac{BC}{4} = \frac{BP + CP}{4}.$$

We get

$$\frac{BP}{CP} = \frac{1+4}{4-1} = \frac{5}{3}.$$

■

Exercise 4 (RMO 1990 P4,). Find the remainder when 2^{1990} is divided by 1990.

Solution 4. Since 199 is a prime, by Fermat's little theorem, we obtain

$$2^{198} \equiv 1 \bmod 199,$$

which yields

$$2^{1990} \equiv 2^{1980} \cdot 2^{10} \bmod 199 \equiv 29 \bmod 199.$$

Using $2^4 \equiv 1 \bmod 5$, we get

$$2^{1990} \equiv 4 \bmod 5 \equiv 29 \bmod 5.$$

Since $5, 199$ are coprime, this shows that

$$2^{1990} \equiv 29 \bmod 199 \cdot 5.$$

Observe that

$$2^{1990} - (29 + 199 \cdot 5)$$

is divisible by $199 \cdot 5$, and also by 2. Since $2, 199 \cdot 5$ are coprime, it follows that

$$2^{1990} \equiv 29 + 199 \cdot 5 \bmod 2 \cdot 199 \cdot 5,$$

which implies that

$$2^{1990} \equiv 1024 \bmod 1990.$$

So, one obtains 1024 as the remainder upon dividing 2^{1990} by 1990. ■

Exercise 5 (RMO 1990 P5,). Let P be a point inside a triangle ABC. Let s denote the semiperimeter $\frac{1}{2}(AB + BC + CA)$. Prove that

$$s < AP + BP + CP < 2s.$$

Solution 5. Applying the triangle inequality to PBC, we get $BC < BP + CP$. Similarly, the inequalities $CA < CP + AP, AB < AP + BP$ follow. Adding them yields

$$AB + BC + CA < 2(AP + BP + CP),$$

or equivalently, $s < AP + BP + CP$ holds. Let D denote the point on BC such that A, P, D are collinear. Note that

$$\begin{aligned} AP + BP &< AP + BD + DP \quad \text{(applying the triangle inequality to } BDP\text{)} \\ &= AD + BD \\ &< AC + DC + BD \quad \text{(applying the triangle inequality to } ADC\text{)} \\ &= AC + BC \end{aligned}$$

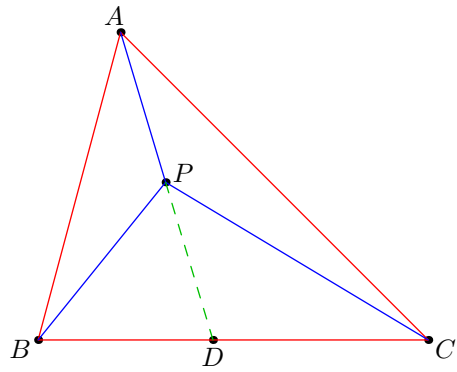


Figure 2: India RMO 1990 P5, Exercise 5

hold. Similarly, it follows that

$$BP + CP < AB + CA, CP + AP < AB + BC.$$

Adding these inequalities, we obtain

$$AP + BP + CP < AB + BC + CA = 2s.$$

■

Exercise 6 (RMO 1990 P6,). N is a 50 digit number (in the decimal scale). All digits except the 26th digit (from the left) are 1. If N is divisible by 13, find the 26th digit.

Solution 6. Let a denote the 26-th digit of N. The given condition yields

$$N = 1 + 10 + 10^2 + \cdots + 10^{49} + (a-1)10^{50-26}.$$

Assume that N is divisible by 13. This gives

$$1 + 10 + 10^2 + \cdots + 10^{49} + (a-1)10^{24} \equiv 0 \bmod 13.$$

Applying Fermat's little theorem, we obtain $10^{12} \equiv 1 \bmod 13$, and this shows that

$$10^{24} \equiv 1 \bmod 13, \quad 10^{50} - 1 \equiv 10^2 - 1 \bmod 13.$$

Note that

$$9(1 + 10 + 10^2 + \cdots + 10^{49}) + 9(a-1)10^{24} \equiv 0 \bmod 13$$

holds, which shows that

$$10^{50} - 1 + 9(a-1)10^{24} \equiv 0 \bmod 13,$$

and this implies that

$$10^2 - 1 + 9(a-1) \equiv 0 \bmod 13.$$

Since the integers $9, 13$ are relatively prime, we obtain

$$11 + a - 1 \equiv 0 \bmod 13.$$

It follows that $a = 3$. ■

References

[**Hun08**] Pham Kim Hung. *Secrets in Inequalities. Volume 1: basic inequalities*. Gil Publishing House, 2008 (cited p. 1)