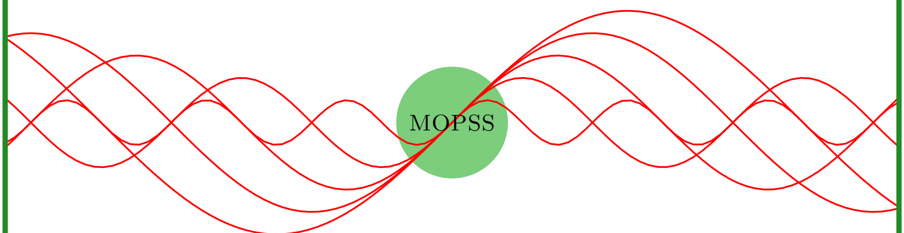
System of equations

MOPSS

14 May 2025

Mathematics Olympiad

Problem Solving Sessions



DEPARTMENT OF MATHEMATICS
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https://jpsaha.github.io/MOTP/MOPSS/

Suggested readings

- Evan Chen's advice On reading solutions, available at https://blog.evanchen.cc/2017/03/06/on-reading-solutions/.
- Evan Chen's Advice for writing proofs/Remarks on English, available at https://web.evanchen.cc/handouts/english/english.pdf.
- Notes on proofs by Evan Chen from OTIS Excerpts [**Che25**, Chapter 1].
- Tips for writing up solutions by Edward Barbeau, available at https://www.math.utoronto.ca/barbeau/writingup.pdf.
- Evan Chen discusses why math olympiads are a valuable experience for high schoolers in the post on Lessons from math olympiads, available at https://blog.evanchen.cc/2018/01/05/lessons-from-math-olympiads/.

List of problems and examples

§1 Systems of equations

§1.1 Factorization

Example 1.1. Find the integral solutions of the equation $y^3 - x^3 = 91$.

> **Walkthrough** — Factorize $y^3 - x^3$, and use the prime factorization of 91. Also note that $x^2 + xy + y^2$ is nonnegative and
> $$y - x \leq y^2 + x^2 \leq x^2 + xy + y^2$$
> holds for any two integers x, y.

Example 1.2. [**WH96**, Problem 14] Let r, s be nonzero integers. Prove that the equation
$$(r^2 - s^2)x^2 - 4rsxy - (r^2 - s^2)y^2 = 1$$
has no solutions in integers.

> **Walkthrough** — Note that
> $$(r^2 - s^2)x^2 - 4rsxy - (r^2 - s^2)y^2 = (rx - sy)^2 - (ry + sx)^2.$$
> Show that
> $$\begin{pmatrix} r & -s \\ s & r \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \pm 1 \\ 0 \end{pmatrix}$$
> holds.

Example 1.3. [**HW97**, Problem 7] Prove that the equation
$$x^4 + y^4 + z^4 - 2x^2y^2 - 2y^2z^2 - 2z^2x^2 = 24$$
has no solution in integers x, y, z.

> **Walkthrough** — Note that
> $$\begin{aligned} &x^4 + y^4 + z^4 - 2x^2y^2 - 2y^2z^2 - 2z^2x^2 \\ &= (x^2 + y^2 - z^2)^2 - (2xy)^2 \\ &= -(x + y + z)(x + y - z)(y + z - x)(z + x - y) \end{aligned}$$
> holds, and any two of the above four factors are of the same parity. Does 2^4 divide 24?

Example 1.4 (India RMO 1992 P1)**.** Determine the set of integers n for which $n^2 + 19n + 92$ is a square of an integer.

Solution 1. Let n be an integer such that $n^2 + 19n + 92 = m^2$ holds for some non-negative integer m. This gives

$$(2n + 19)^2 + 7 = (2m)^2,$$

which yields

$$7 = (2m - 2n - 19)(2m + 2n + 19).$$

Since $2m+2n+19$ is positive, it follows that $(2m-2n-19, 2m+2n+19)$ is equal to $(7, 1)$ or $(1, 7)$. This shows that (m, n) is equal to one of $(2, -11), (2, -8)$, and consequently, n is equal to one of $-11, -8$. Note that

$$(-11)^2 - 19 \cdot 11 + 92 = 4, \quad (-8)^2 - 19 \cdot 8 + 92 = 4$$

holds. This proves that the integers satisfying the given condition are precisely $-11, -8$. ■

Example 1.5 (India RMO 2001 P2)**.** Find all primes p, q such that $p^2+7pq+q^2$ is a perfect square.

Solution 2. Let p, q be primes such that

$$p^2 + 7pq + q^2 = m^2 \tag{1}$$

holds for some positive integer m. Note that m is congruent to one of $q, -q$ modulo p. Write $m = kp \pm q$ for some integer k. Substituting $m = kp \pm q$ in Eq. (1) yields

$$p(k^2 - 1) = (7 \mp 2k)q. \tag{2}$$

Let us consider the case that $p > q$. This gives $m^2 \geq 9q^2$, and hence we obtain $k > 1$. This implies that $k^2 - 1$ is positive, and hence, so is $7 \pm 2k$. Using $p > q$, we get

$$k^2 - 1 < 7 \mp 2k,$$

which yields $(k \pm 1)^2 < 9$. Noting that k is positive, it follows that $k \pm 1$ is equal to one of $0, 1, 2$, and hence k is equal to one of $2, 3$. Substituting $k = 2$ in Eq. (2) yields $3p = (7 \mp 4)q$. Since $p > q$, we obtain $3p = 11q$, which shows that $p = 11, q = 3$. Substituting $k = 3$ in Eq. (2) yields $8p = (7 \mp 6)q$, which implies that $8p = q$ or $8p = 13q$, which is impossible. It follows that any pair of primes (p, q) such that $p^2 + 7pq + q^2$ is a perfect square and $p > q$ holds, is equal to $(11, 3)$. Note that

$$11^2 + 7 \cdot 11 \cdot 3 + 3^2 = 121 + 231 + 9 = 361 = 19^2$$

holds. We conclude that $(p, q) = (11, 3)$ is the only solution when $p > q$.

Since $p^2 + 7pq + q^2$ is symmetric in p, q, it follows that $(p, q) = (3, 11)$ is the only solution when $p < q$.

Also note that if $p = q$, then $p^2 + 7pq + q^2 = (3p)^2$ is a perfect square. So the solutions are are precisely $(3, 11), (11, 3)$, and the tuples of the form (r, r), where r runs over the set of primes. ■

Example 1.6 (India Pre-RMO 2012 P5)**.** Let $S_n = n^2 + 20n + 12$, n a positive integer. What is the sum of all possible values of n for which S_n is a perfect square?

Solution 3. Let n be a positive integer such that

$$n^2 + 20n + 12 = m^2$$

holds for some positive integer m. Completing squares, we obtain

$$(n + 10)^2 - m^2 = 88,$$

which yields

$$(n + 10 + m)(n + 10 - m) = 88.$$

Note that at least one of the integers $n + 10 + m, n + 10 - m$ is even, and they are of the same parity. This shows that $(n + 10 + m, n + 10 - m)$ is equal to one of

$$(44, 2), (22, 4).$$

It follows that $(n + 10, m)$ is equal to one of

$$(23, 21), (13, 9),$$

and consequently, (n, m) is equal to one of

$$(13, 21), (3, 9).$$

Note that

$$13^2 + 20 \cdot 13 + 12 = 169 + 260 + 12 = 441 = 21^2,$$

and

$$3^2 + 20 \cdot 3 + 12 = 9 + 60 + 12 = 81 = 9^2$$

hold. We conclude that the sum of all possible values of n for which S_n is a perfect square is equal to $13 + 3 = 16$. ■

§1.2 Completing squares

Example 1.7. Find all solutions of $x^2 + 3y^2 = 4$ in integers. Use it to find all solutions of $m^2 + mn + n^2 = 1$ in integers.

Walkthrough — Observe that

$$4(m^2 + mn + n^2) = (2m + n)^2 + 3n^2.$$

Example 1.8 (India BStat-BMath 2013 P7)**.** Let N be a positive integer such that $N(N-101)$ is the square of a positive integer. Find all possible values of N. (Note that 101 is a prime number).

Solution 4. Let N be a positive integer such that for some positive integer k,

$$N(N-101) = k^2$$

holds. This gives

$$(2N-101)^2 = 4k^2 + 101^2,$$

which yields

$$(2N-101-2k)(2N-101+2k) = 101^2.$$

Using $N(N-101) = k^2$, $N \geq 1$ and $k \geq 1$, it follows that $N > 101$, which shows that

$$2N - 101 + 2k = N - 101 + N + 2k > 101.$$

Since 101 is a prime, we obtain that $(2N-101-2k, 2N-101+2k)$ is equal to $1, 101^2$, which yields

$$N = \frac{1 + 101^2 + 2 \cdot 101}{4} = \frac{102^2}{4} = 51^2 = 2601.$$

Note that

$$2601(2601-101) = 51^2 \times 50^2,$$

which is a perfect square. This proves that $N = 2601$ is a only solution. ■

§1.3 Arrange in Order

Example 1.9 (India RMO 1996 P2)**.** Find all triples (a, b, c) of positive integers such that

$$\left(1 + \frac{1}{a}\right)\left(1 + \frac{1}{b}\right)\left(1 + \frac{1}{c}\right) = 3.$$

Solution 5. Let a, b, c be positive integers, satisfying the above equation. Since

$$\left(1 + \frac{1}{a}\right)\left(1 + \frac{1}{b}\right)\left(1 + \frac{1}{c}\right)$$

is symmetric with respect to a, b, c, it suffices to consider the case $a \geq b \geq c$. Note that

$$3 = \left(1 + \frac{1}{a}\right)\left(1 + \frac{1}{b}\right)\left(1 + \frac{1}{c}\right) \leq \left(1 + \frac{1}{c}\right)^3$$

holds. If $c \geq 3$ holds, then we would obtain $3 \leq (1 + \frac{1}{c})^3$ implying $81 \leq 64$, which is impossible. This shows that $c = 1$ or $c = 2$.

Let us consider the case that $c = 2$. Then

$$\left(1+\frac{1}{a}\right)\left(1+\frac{1}{b}\right) = 2$$

holds, which shows that $2 \leq (1+\frac{1}{b})^2$. This gives $b < 3$. Note that $b = 1, c = 2$ is not possible. This yields $(a, b, c) = (3, 2, 2)$.

Now, let us consider the case that $c = 1$. It follows that

$$\left(1+\frac{1}{a}\right)\left(1+\frac{1}{b}\right) = \frac{3}{2},$$

which implies $\frac{3}{2} \leq (1+\frac{1}{b})^2$, and hence, we get $b < 5$. It follows that $b \neq 1$ and $b \neq 2$. Consequently, we obtain that (a, b, c) is equal to one of $(8, 3, 1), (5, 4, 1)$.

Note that the triples

$$(3, 2, 2), (8, 3, 1), (5, 4, 1)$$

also satisfy the given equation. It follows that these are precisely all the solutions of the given equation in the positive integers under the hypothesis that $a \geq b \geq c$.

Since $(1+\frac{1}{a})(1+\frac{1}{b})(1+\frac{1}{c})$ is symmetric in a, b, c, the required solutions are obtained by permuting the coordinates of these three solutions, that, the required solutions are precisely

$$(3, 2, 2), (2, 3, 2), (2, 2, 3),$$

$$(8, 3, 1), (8, 1, 3), (3, 8, 1), (3, 1, 8), (1, 8, 3), (1, 3, 8),$$

$$(5, 4, 1), (5, 1, 4), (4, 5, 1), (4, 1, 5), (1, 4, 5), (1, 5, 4).$$

■

Example 1.10 (India RMO 2010 P4)**.** Find three distinct positive integers with the least possible sum such that the sum of the reciprocals of any two integers among them is an integral multiple of the reciprocal of the third integer.

Solution 6. Let a, b, c be distinct positive integers with the least possible sum such that

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c} = \frac{p}{a} = \frac{q}{b} = \frac{r}{c} \tag{3}$$

holds, where p, q, r are positive integers. Since a, b, c are positive, it follows that p, q, r are greater than 1. By reordering a, b, c if necessary, we may and do assume that $a < b < c$, or equivalently, $p < q < r$ holds. Note that

$$\frac{3}{a} > \frac{1}{a}+\frac{1}{b}+\frac{1}{c} = \frac{p}{a},$$

which gives $p = 2$. This yields

$$\frac{1}{a} = \frac{1}{b} + \frac{1}{c} < \frac{2}{b},$$

which implies that $q < 2p = 4$, and using $p < q$, we obtain $q = 3$. Now Eq. (3) gives

$$2\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) = \frac{p}{a} + \frac{q}{b},$$

and hence, we obtain $\frac{2}{c} = \frac{1}{b}$, which yields $r = 2q = 6$. This shows that $a : b : c$ is equal to $2 : 3 : 6$. Noting that $2, 3, 6$ have no common divisor larger than 1, it follows that $a + b + c$ is a multiple of $2 + 3 + 6 = 11$. Note that

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{6} = 1,$$

which is an integral multiple of the reciprocal of any of $2, 3, 6$. This proves that $2, 3, 6$ are three distinct positive integers with the least possible sum such that the sum of the reciprocals of any two integers among them is an integral multiple of the reciprocal of the third integer. Moreover, this proof also shows that these are unique up to reordering. ■

§1.4 Using bounds

Example 1.11 (Canada CMO 1983 P1)**.** Find all positive integers w, x, y and z which satisfy $w! = x! + y! + z!$.

Solution 7. Let w, x, y, z be positive integers satisfying the given equation. Since $x! + y! + z!$ is symmetric in x, y, z, it suffices to consider the case that $x \leq y \leq z$. Note that $y - x \leq 1$, otherwise, if $y \geq x + 2$, then we would have

$$\frac{w!}{x!} = 1 + \frac{y!}{x!} + \frac{z!}{x!},$$

which is impossible since

$$\frac{w!}{x!}, \frac{y!}{x!}, \frac{z!}{x!}$$

are even.

Let us consider the case that $y = x$. Then $z \leq x + 2$ holds, otherwise,

$$\frac{w!}{x!} = 2 + \frac{z!}{x!}$$

would not hold, since $\frac{w!}{x!}, \frac{z!}{x!}$ are multiples of 3. If $z = x$, then

$$w \leq \frac{w!}{x!} = 3,$$

which yields that $x = y = z = 2$ and $w = 3$. If $z = x + 1$, then $w > (x + 1)!$ holds, which implies $w \geq x + 2$, and this yields

$$(x + 1)(x + 2) \leq \frac{w!}{x!} = 2 + x + 1,$$

which gives $(x + 1)^2 \leq 2$, which holds for no positive integer x. If $z = x + 2$, then similarly, we obtain

$$(x + 1)(x + 2)(x + 3) \leq \frac{w!}{x!} = 2 + (x + 1)(x + 2),$$

which implies that

$$(x + 1)(x + 2)^2 \leq 2,$$

which holds for no positive integer x.

Now let us consider the case that $y = x + 1$. We obtain $w \geq z + 1$, and this gives

$$\begin{aligned} & x + 2 + \prod_{x+1 \leq t \leq z} t \\ &= \frac{w!}{x!} \\ &= \prod_{x+1 \leq t \leq w} t \\ &= \prod_{x+1 \leq t \leq z} t + \left(\prod_{z+1 \leq s \leq w} s - 1 \right) \prod_{x+1 \leq t \leq z} t, \end{aligned}$$

which yields

$$\begin{aligned} x + 2 &= \left(\prod_{z+1 \leq s \leq w} s - 1 \right) \prod_{x+1 \leq t \leq z} t \\ &\geq (z + 1 - 1) \prod_{x+1 \leq t \leq z} t \\ &\geq y \prod_{x+1 \leq t \leq z} t \\ &\geq (x + 1) \prod_{x+1 \leq t \leq z} t \\ &\geq (x + 1)^2, \end{aligned}$$

which is impossible.

This proves that

$$x = y = z = 2, w = 3$$

is the only solution to the given equation over the positive integers. ■

Remark. After obtaining the above solution, one may easily arrive at the following argument, which is much shorter.

Let x, y, z, w be positive integers satisfying $w! = x! + y! + z!$. Note that $w \geq \max\{x, y, z\} + 1$ holds, which gives

$$w \times \max\{x, y, z\}! \leq w! = x! + y! + z! \leq 3 \times \max\{x, y, z\}!$$

implying $w \leq 3$. It follows that $x = y = z = 2, w = 3$ is the only solution of the given equation.

Example 1.12 (India RMO 2005 P6)**.** Determine all triples (a, b, c) of positive integers such that $a \leq b \leq c$ and

$$a + b + c + ab + bc + ca = abc + 1.$$

Solution 8. Let a, b, c be integers satisfying the given conditions. Note that the above equation can be rewritten as

$$(1 + a)(1 + b)(1 + c) = 2(abc + 1). \tag{4}$$

Note that if the inequalities

$$2^{\frac{1}{3}}a \geq a + 1, 2^{\frac{1}{3}}b \geq b + 1, 2^{\frac{1}{3}}c \geq c + 1$$

hold, then

$$(1 + a)(1 + b)(1 + c) \leq 2abc < 2(abc + 1)$$

holds, which is impossible. Also note that if $a \geq 4$, then

$$\frac{1}{2^{\frac{1}{3}} - 1} = 2^{\frac{2}{3}} + 2^{\frac{1}{3}} + 1 < \frac{8}{5} + \frac{7}{5} + 1 = 4$$

holds, which implies that

$$2^{\frac{1}{3}}a \geq a + 1, 2^{\frac{1}{3}}b \geq b + 1, 2^{\frac{1}{3}}c \geq c + 1.$$

This proves that $a \leq 3$. From Eq. (4), it follows that $a \neq 1$, and hence, a is equal to one of $2, 3$.

Also note that Eq. (4) does not hold if the inequalities

$$\sqrt{2a} \cdot b \geq \sqrt{1 + a}(b + 1), \sqrt{2a} \cdot c \geq \sqrt{1 + a}(c + 1)$$

hold. Observe that if

$$b \geq \begin{cases} 7 & \text{if } a = 2, \\ 5 & \text{if } a = 3, \end{cases}$$

holds, then we obtain

$$\frac{\sqrt{1+a}}{\sqrt{2a}-\sqrt{1+a}} = \begin{cases} \dfrac{\sqrt{3}}{2-\sqrt{3}} < 7 & \text{if } a = 2, \\ \dfrac{2}{\sqrt{6}-2} < 5 & \text{if } a = 3. \end{cases}$$

This shows that $b \leq 6$ if $a = 2$, and $b \leq 4$ if $a = 3$. Note that Eq. (4) is equivalent to

$$(2ab - (1+a)(1+b))c = (1+a)(1+b) - 2,$$

which gives

$$c(b-3) = 3b + 1 \text{ if } a = 2,$$
$$c(b-2) = 2b + 1 \text{ if } a = 3.$$

It follows that if $a = 2$, then $b \neq 2$ and $b \neq 3$, and if $a = 3$, then $b \neq 4$. This shows that (a, b, c) is equal to one of

$$(2, 4, 13), (2, 5, 8), (3, 3, 7).$$

Note that these triples satisfy Eq. (4). Consequently, the above triples are precisely all the solutions. ■

Remark. Note that Eq. (4) suggests to substitute

$$x = 1 + a, y = 1 + b, z = 1 + c,$$

which yields

$$xyz = 2 + 2(x-1)(y-1)(z-1),$$

which can be rewritten as

$$xyz + 2(x + y + z) = 2(xy + yz + zx).$$

The above reduces to

$$2(x+y+z) + xy\left(\frac{z}{3} - 2\right) + yz\left(\frac{x}{3} - 2\right) + zx\left(\frac{y}{3} - 2\right) = 0,$$

which shows that $x \leq 5$. Can one use the above to determine all the solutions?

Example 1.13 (India RMO 2012f P5)**.** Determine all positive integers a, b, c such that $\frac{1}{a} + \frac{2}{b} + \frac{3}{c} = 1$, $a \leq b \leq c$ and a is a prime.

Solution 9. Let a, b, c be integers satisfying the given conditions. Since

$$1 = \frac{1}{a} + \frac{2}{b} + \frac{3}{c} \leq \frac{6}{a}$$

holds, we obtain $a \leq 6$. Since a is a prime, it is equal to one of $2, 3, 5$. Note that

$$\frac{2}{b} < \frac{2}{b} + \frac{3}{c} \leq \frac{5}{b}$$

holds, which implies that

$$\frac{2}{b} < 1 - \frac{1}{a} \leq \frac{5}{b},$$

which yields that

$$\frac{2}{1-\frac{1}{a}} < b \leq \frac{5}{1-\frac{1}{a}}.$$

It follows that $5 \leq b \leq 10$ if $a = 2$, $4 \leq b \leq 7$ if $a = 3$, and $b \leq 6$ if $a = 5$. Note that if (a, b) is equal to

$$(2, 9), (3, 5), (3, 7), (5, 5), (5, 6),$$

then no integer c satisfies the given equation. This shows that (a, b, c) is equal to one of

$$(2, 5, 30), (2, 6, 18), (2, 7, 14), (2, 8, 12), (2, 10, 10), (3, 4, 18), (3, 6, 9).$$

Noting that the above triples satisfy the given equation, we conclude that the required solutions are precisely the ones above. ■

Example 1.14 (India RMO 2012c P6)**.** Find all positive integers such that $3^{2n} + 3n^2 + 7$ is a perfect square.

Solution 10. Let n, m be positive integers such that

$$3^{2n} + 3n^2 + 7 = m^2$$

holds. This implies

$$3n^2 + 7 = (m - 3^n)(m + 3^n).$$

Note that any two positive integers a, b satisfy $ab > a - b$. Since $m - 3^n, m + 3^n$ are positive, it follows that

$$3n^2 + 7 > 2 \cdot 3^n.$$

Applying the binomial theorem, we obtain

$$3^n \geq 1 + \binom{n}{1} 2 + \binom{n}{2} 2^2 \geq 1 + 2n + 2^2 \frac{n(n-1)}{2} = 1 + 2n^2,$$

which yields

$$2(1 + 2n^2) < 3n^2 + 7,$$

which is equivalent to $n^2 < 5$. This shows that n is equal to 1 or 2. Note that $3^{2n} + 3n^2 + 7$ is equal to 19 (resp. 100) for $n = 1$ (resp. $n = 2$). So the required solution is $n = 2$. ■

Example 1.15 (India RMO 2015b P6)**.** Find the number of integers m that satisfy both the following properties:

1. $1 \leq m \leq 5000$,
2. $[\sqrt{m}] = [\sqrt{m+125}]$.

Solution 11. If m is positive integer satisfying $[\sqrt{m}] = [\sqrt{m+125}]$, then for $k = [\sqrt{m}]$, we have $k \leq \sqrt{m} < \sqrt{m+125} < k+1$, which yields $k^2 \leq m < m+125 < (k+1)^2$, and this implies that

$$2k+1 = (k+1)^2 - k^2 \geq m + 126 - m = 126,$$

or equivalently, $k \geq 63$ holds. This shows that for any positive integer m satisfying $[\sqrt{m}] = [\sqrt{m+125}]$, we have $k^2 \leq m \leq (k+1)^2 - 126$ for some integer $k \geq 63$.

Conversely, if m is an integer satisfying $k^2 \leq m \leq (k+1)^2 - 126$ for some positive integer $k \geq 63$, then we obtain $k^2 \leq m < m+125 < (k+1)^2$, which gives $k \leq \sqrt{m} < \sqrt{m+125} < k+1$, and consequently, $[\sqrt{m}] = [\sqrt{m+125}]$ holds.

So the number of integers satisfying the given conditions is equal to the number of positive integers m satisfying $m \leq 5000$ and $k^2 \leq m \leq (k+1)^2 - 126$ for some positive integer $k \geq 63$. Note that any such integer k satisfies $k^2 \leq m \leq 5000 < 71^2$, which gives $k \leq 70$. Moreover, if ℓ is an integer satisfying $63 \leq \ell \leq 70$ and an integer m satisfies $\ell^2 \leq m \leq (\ell+1)^2 - 126$, then using

$$(\ell+1)^2 - 126 \leq 71^2 - 126 = 4900 + 140 + 1 - 126 < 5000,$$

it follows that $m \leq 5000$. This proves that the integers satisfying the given conditions are precisely the integers m satisfying $k^2 \leq m \leq (k+1)^2 - 126$ for some integer $63 \leq k \leq 70$. Hence, the required number is equal to

$$\begin{aligned} &\sum_{k=63}^{70} ((k+1)^2 - 126 - k^2 + 1) \\ &= \sum_{k=63}^{70} (2k - 124) \\ &= 2 + 4 + 6 + \cdots + 16 \\ &= 72. \end{aligned}$$

■

§1.5 Warm up

Example 1.16 (India RMO 1999 P6)**.** Find all solutions in integers m, n of the equation

$$(m-n)^2 = \frac{4mn}{m+n-1}.$$

Solution 12. Let m, n be integers satisfying $m+n \neq 1$ and the above equation. Note that the above equation is equivalent to

$$(m-n)^2 = \frac{(m+n)^2}{m+n} = m+n,$$

which holds if and only if

$$\frac{1}{2}(m-n)(m-n-1) = n.$$

Writing $m-n=k$, it follows that (m,n) is equal to

$$\left(k + \frac{1}{2}k(k-1), \frac{1}{2}k(k-1)\right) = \left(\frac{1}{2}k(k+1), \frac{1}{2}k(k-1)\right).$$

Also note that for any integer k, the pair

$$\left(\frac{1}{2}k(k+1), \frac{1}{2}k(k-1)\right)$$

is a solution to the given equation if $k^2 \neq 1$. This shows that the solution is

$$\left\{\left(\frac{1}{2}k(k+1), \frac{1}{2}k(k-1)\right) \mid k \in \mathbb{Z} \setminus \{\pm 1\}\right\}.$$

■

Example 1.17 (India RMO 2007 P2)**.** Let a, b, c be three natural numbers such that $a < b < c$ and $\gcd(c-a, c-b) = 1$. Suppose there exists an integer d such that $a+d, b+d, c+d$ form the sides of a right-angled triangle. Prove that there exist integers ℓ, m such that $c+d = \ell^2 + m^2$.

Solution 13. Since $a < b < c$, and the integers $a+d, b+d, c+d$ form the sides of a right-angled triangle, it follows that

$$(c+d)^2 = (a+d)^2 + (b+d)^2. \tag{5}$$

Writing the above as a quadratic equation d, we obtain

$$d^2 + 2(a+b-c)d + a^2 + b^2 - c^2 = 0,$$

which implies that

$$d = (c-a-b) \pm \sqrt{(a+b-c)^2 - (a^2+b^2-c^2)}.$$

Since $a+b+d-c = (a+d)+(b+d)-(c+d)$ is positive, we obtain

$$\begin{aligned} d &= (c-a-b) + \sqrt{(a+b-c)^2 - (a^2+b^2-c^2)} \\ &= (c-a-b) + \sqrt{2c^2+2ab-2bc-2ca} \\ &= (c-a-b) + \sqrt{2(c-a)(c-b)}. \end{aligned}$$

Since d is an integer, it follows that $2(c-a)(c-b)$ is a perfect square. Note that the integers $c-a, c-b$ are positive and relatively prime. So there are positive integers m, n such that $c-a, c-b$ are equal to $2m^2, n^2$ in some order. This yields

$$\begin{aligned} c+d &= (c-a)+(c-b)+\sqrt{2(c-a)(c-b)} \\ &= 2m^2+n^2+2mn \\ &= m^2+(m+n)^2, \end{aligned}$$

which proves the result. ■

Remark. After obtaining Eq. (5), one may also argue as follows. Note that Eq. (5) implies that a divisor of any two of $a+d, b+d, c+d$, divides all of them, and hence also divides the differences $c-a, c-b$, which are coprime. This shows that $a+d, b+d, c+d$ are pairwise coprime integers, and $(a+d, b+d, c+d)$ is a primitive Pythagorean triple. It follows (**how?**) that $c+d$ is equal to ℓ^2+m^2 for some integers ℓ, m.

It is **not a good idea** to apply the classification of primitive Pythagorean triples to conclude that $c+d$ is the sum of two squares, since the above solution of Example 1.17 is **no different from** (one proof of) the classification of primitive Pythagorean triples, which is provided below.

A careful reading of the above solution of Example 1.17 leads to the following proof of the primitive Pythagorean triples as follows.

Lemma 1 (Classification of primitive Pythagorean triples)
Let $1 \leq x \leq y \leq z$ be integers satisfying

$$x^2 + y^2 = z^2.$$

Then the following statements are equivalent.

(i) Some two of x, y, z are relatively prime.

(ii) Any two of x, y, z are relatively prime.

(iii) The integers $z - x, z - y$ are relatively prime.

If any of the above conditions holds, then there are relatively prime positive integers $a > b$ such that x, y are equal to $a^2 - b^2, 2ab$ in some order, and z is equal to $a^2 + b^2$.

For a **geometric proof** of the above, we refer to [**ST15**, §1.1].

Proof. Note that

$$x^2 = (z - y)(z + y), \quad y^2 = (z - x)(z + x)$$

holds. So any common prime divisor of $z - x, z - y$ is also a common divisor of x, y. Using $x^2 + y^2 = z^2$, it follows that any common divisor of x, y is also a common divisor of $z - x, z - y$. The equivalence of the three statements follows.

Assume that one of the given conditions holds. Note that

$$\begin{aligned}(x + y - z)^2 &= (x + y - z)^2 - (x^2 + y^2 - z^2) \\ &= 2z^2 + 2xy - 2z(x + y) \\ &= 2(z - x)(z - y)\end{aligned}$$

holds. Since $z - x, z - y$ are relatively prime, and $2(z - x)(z - y)$ is a perfect square, it follows that there are positive integers m, n such that $z - x, z - y$ are equal to $2m^2, n^2$ in some order. Note that

$$\begin{aligned}z &= (z - x) + (z - y) + (x + y - z) \\ &= 2m^2 + n^2 + 2mn \\ &= m^2 + (m + n)^2,\end{aligned}$$

and this implies that

$$\begin{aligned}z - 2m^2 &= (m + n)^2 - n^2, \\ z - n^2 &= 2m(m + n).\end{aligned}$$

Putting

$$a = m + n, b = m,$$

it follows that x, y are equal to $a^2 - b^2, 2ab$ in some order, and z is equal to $a^2 + b^2$. Since x, y are relatively prime, we get that the integers a, b are relatively prime. □

Example 1.18 (India RMO 2008 P2)**.** Prove that there exist two infinite sequences $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$ of positive integers such that the following conditions hold simultaneously:

(i) $0 < a_1 < a_2 < a_3 < \cdots$,

(ii) $a_n < b_n < a_n^2$ for all $n \geq 1$,

(iii) $a_n - 1$ divides $b_n - 1$ for all $n \geq 1$,

(iv) $a_n^2 - 1$ divides $b_n^2 - 1$ for all $n \geq 1$.

Solution 14. We claim that it suffices to prove that for any positive integer N, there exist integers $a, b > N$ such that $a < b < a^2$, and $a - 1$ divides $b - 1$, $a^2 - 1$ divides $b^2 - 1$. Indeed, if this statement is true, then there exist positive integers a_1, b_1 such that $a_1 < b_1 < a_1^2$, and $a_1 - 1$ divides $b_1 - 1$, $a_1^2 - 1$ divides $b_1^2 - 1$. Moreover, if for some positive integer $n \geq 1$, there are positive integers $a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n$ such that $a_1 < a_2 < \cdots < a_n, b_1 < b_2 < \cdots < b_n$, and $a_k - 1$ divides $b_k - 1$ and $a_k^2 - 1$ divides $b_k^2 - 1$ for any $1 \leq k \leq n$, then by the above statement, there exist integers a, b such that $a, b > a_n + b_n$, $a < b < a^2$, and $a - 1$ divides $b - 1$, $a^2 - 1$ divides $b^2 - 1$, then one can define $a_{n+1} = a, b_{n+1} = b$. Applying induction, we obtain two infinite sequences $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$ as desired. Now it remains to prove that for any positive integer N, there exist integers $a, b > N$ such that $a < b < a^2$, and $a - 1$ divides $b - 1$, $a^2 - 1$ divides $b^2 - 1$.

Note that if a, b are two integers such that $a - 1$ divides $b - 1$, then $b = 1 + (a - 1)k$ for some integer k, and hence

$$\begin{aligned} b^2 - 1 &= (b - 1)(b + 1) \\ &= k(a - 1)((a - 1)k + 2) \\ &= k(a - 1)((a + 1)k - (2k - 2)) \\ &= k^2(a^2 - 1) - 2k(k - 1)(a - 1) \end{aligned}$$

holds, which shows that $b^2 - 1$ is divisible by $a^2 - 1$ if $a + 1$ divides $2k$, which holds if a is odd and $k = \frac{a+1}{2}$. In fact, for any positive integer N, setting

$$a = 2N + 1, \quad b = 1 + (a - 1)\frac{a + 1}{2} = \frac{a^2 + 1}{2} = 2N^2 + 2N + 1,$$

the inequality $a < b < a^2$ follows, and $a - 1 = 2N$ divides $b - 1 = 2N(N + 1)$, $a^2 - 1 = 4N(N + 1)$ divides $b^2 - 1 = 4N(N + 1)(N^2 + N + 1)$. This completes the proof. ■

Remark. The above argument (that is, **only after having** the above argument, it) shows that any strictly increasing sequence $\{a_n\}_{n\geq 1}$ of odd positive integers with $a_1 \geq 3$, and the $\{b_n\}_{n\geq 1}$, defined by

$$b_n = \frac{a_n^2+1}{2} \quad \text{for any integer } n \geq 1,$$

have the required properties.

Example 1.19 (India RMO 2008 P5)**.** Three nonzero real numbers a, b, c are said to be in harmonic progression if $\frac{1}{a} + \frac{1}{c} = \frac{2}{b}$. Find all three term harmonic progressions a, b, c of strictly increasing positive integers in which $a = 20$ and b divides c.

Solution 15. Let b, c be positive integers such that $20 < b < c$ holds, and b divides c and

$$\frac{1}{20} + \frac{1}{c} = \frac{2}{b}$$

holds. For some positive integer $k > 1$, we have $c = bk$, which yields

$$\frac{1}{20} + \frac{1}{bk} = \frac{2}{b},$$

and this gives

$$bk = 20(2k-1).$$

Since $k, 2k-1$ are relatively prime and b is an integer, it follows that k divides 20, and hence, k is equal to one of $2, 4, 5, 10$ or 20. This shows that (a, b, c) is equal to one of

$$(20, 30, 60), (20, 35, 140), (20, 36, 180), (20, 38, 380), (20, 39, 780).$$

Note that any of the above triples satisfy the required conditions. This proves that the above ones are all the three term harmonic progressions satisfying the required conditions. ∎

Example 1.20 (India RMO 2010 P6)**.** For each integer $n \geq 1$, define

$$a_n = \left[\frac{n}{[\sqrt{n}]}\right],$$

where $[x]$ denotes the largest integer not exceeding x, for any real number x. Find the number of all n in the set $\{1, 2, 3, \dots, 2010\}$ for which $a_n > a_{n+1}$.

Solution 16. Let n be a positive integer such that $a_n > a_{n+1}$ holds. Note that $[\sqrt{n}], [\sqrt{n+1}]$ are not equal, otherwise, we would obtain

$$\frac{n+1}{[\sqrt{n+1}]} = \frac{n+1}{[\sqrt{n}]} \geq \frac{n}{[\sqrt{n}]},$$

implying $a_{n+1} \geq a_n$. Let k denote the largest integer satisfying $k^2 \leq n$. In other words, k denotes the integer $[\sqrt{n}]$. Since $[\sqrt{n}], [\sqrt{n+1}]$ are not equal, we obtain $[\sqrt{n+1}] > k$, which gives $[\sqrt{n+1}] \geq k+1$. This shows that $n+1 \geq (k+1)^2$. Since k is the largest integer satisfying $k^2 \leq n$, we obtain $n+1 = (k+1)^2$. We conclude that if $a_n > a_{n+1}$ holds, then $n+1$ is a perfect square. Also note that the converse of this statement holds, that is, if $n+1$ is a perfect square, then $a_n > a_{n+1}$ holds. Indeed, if $n+1 = m^2$ holds for some positive integer m, then it follows that

$$\frac{n}{[\sqrt{n}]} - \frac{n+1}{[\sqrt{n+1}]} = \frac{m^2-1}{m-1} - \frac{m^2}{m} = 1,$$

which yields

$$\left[\frac{n}{[\sqrt{n}]}\right] \geq \left[\frac{n+1}{[\sqrt{n+1}]}\right] \geq 1 > 0.$$

This proves that for a positive integer n, the inequality $a_n > a_{n+1}$ holds if and only if $n+1$ is a perfect square. Noting that

$$45^2 - 1 = 2025 - 1 > 2010, 44^2 = 2025 + 1 - 90 < 2010$$

holds, we conclude that the positive integers n in the set $\{1, 2, \ldots, 2010\}$ satisfying $a_n > a_{n+1}$ are precisely

$$2^2 - 1, 3^2 - 1, \ldots, 44^2 - 1.$$

So there are 43 values of n satisfying the given condition. ■

Example 1.21 (India RMO 2012e P3)**.** Find all natural numbers x, y, z such that

$$(2^x - 1)(2^y - 1) = 2^{2^z} + 1.$$

Solution 17. Let x, y, z be natural numbers satisfying the above equation. Note that the above equation can be rewritten as

$$2^{x+y} = 2^x + 2^y + 2^{2^z}.$$

Let us first consider the case that $x \leq y$. Note that $2^z \geq x$ holds and we have

$$2^y = 1 + 2^{y-x} + 2^{2^z - x}. \tag{6}$$

Let us consider the case that $x = y$. Then we obtain

$$2^y = 2 + 2^{2^z - x},$$

which gives $y = 2, 2^z - x = 1$, which yields

$$2^z = x + 1 = y + 1 = 3,$$

which is impossible. This shows that $x < y$.

Let us consider the case that $x < y$. Using Eq. (6), we obtain $2^z = x$, $y - x = 1$ and $y = 2$, which gives

$$x = 1, y = 2, z = 0.$$

It follows that any solution of the given equation in the natural numbers satisfying $x \leq y$ is equal to $(1, 2, 0)$. Since the given equation is symmetric in x, y, it follows that any given solution of this equation is equal to one of

$$(1, 2, 0), (2, 1, 0).$$

Note that the above triples satisfy the given equation. Hence, the above triples are precisely all the solutions of the given equation in the natural numbers. ■

Example 1.22 (India RMO 2013a P3)**.** Find all primes p and q such that p divides $q^2 - 4$ and q divides $p^2 - 1$.

Solution 18. Let p, q be primes such that p divides $q^2 - 4$ and q divides $p^2 - 1$. If p is equal to 2, then using that p divides $q^2 - 4$, it follows that $q = 2$, which is impossible since q divides $p^2 - 1$. This shows that p is odd.

Since p divides $q^2 - 4$, it follows that p divides $q - 2$, or p divides $q + 2$. Using the hypothesis that q divides $p^2 - 1$, we obtain q divides at least one of $p - 1, p + 1$, and hence, $q \leq p + 1$ holds. If p divides $q - 2$, then using $q - 2 \leq p - 1$, we obtain that $q - 2 = 0$, that is, $q = 2$. If p divides $q + 2$, then using $0 < q + 2 \leq p + 3$, we obtain $q + 2 = p$, and using that q divides $p^2 - 1 = (q+2)^2 - 1 = q^2 + 4q + 3$, we get $q = 3$, and hence, $p = 5$. This proves that (p, q) is equal to $(5, 3)$, or that p is odd, and $q = 2$.

Note that if (p, q) is equal to $(5, 3)$, then the required divisibility conditions hold. Moreover, these conditions are also satisfied if p is an odd prime and $q = 2$.

It follows that the required pairs of primes (p, q) are precisely the elements of

$$\{(5, 3)\} \cup \{(r, 2) \,|\, r \text{ is an odd prime}\}.$$

■

Example 1.23 (India RMO 2013e P2)**.** Find all triples (p, q, r) of primes such that $pq = r + 1$ and $2(p^2 + q^2) = r^2 + 1$.

Solution 19. Let p, q, r be primes satisfying the above conditions. Since 2 divides $r^2 + 1$, it follows that r is odd. This shows that pq is even, and hence at least one of p, q is even.

Let us consider the case that $p = 2$. We obtain $q = \frac{r+1}{2}$. This yields

$$2\left(2^2 + \frac{(r+1)^2}{2^2}\right) = r^2 + 1,$$

which implies

$$r^2 - 2r - 15 = 0,$$

which shows that $r = 5$, and hence, (p, q, r) is equal to $(2, 3, 5)$. Since the given equations are symmetric in p, q, it follows that if $q = 2$, then (p, q, r) is equal to $(3, 2, 5)$. Note that the triples $(2, 3, 5)$ and $(3, 2, 5)$ satisfy the given equations.

We conclude that $(2, 3, 5), (3, 2, 5)$ are precisely all the triples of primes satisfying the given conditions. ■

Example 1.24 (India RMO 2013a P4)**.** Find the number of 10-tuples $(a_1, a_2, \ldots, a_9, a_{10})$ of integers such that $|a_1| \leq 1$ and

$$a_1^2 + a_2^2 + a_3^2 + \cdots + a_{10}^2 - a_1a_2 - a_2a_3 - a_3a_4 - \cdots - a_9a_{10} - a_{10}a_1 = 2.$$

Solution 20. Let $(a_1, \ldots, a_{10})$ be a tuple of integers satisfying $|a_1| \leq 1$, and the above equation. Note that the above equation can be rewritten as

$$(a_1 - a_2)^2 + (a_2 - a_3)^2 + \cdots + (a_9 - a_{10})^2 + (a_{10} - a_1)^2 = 4.$$

Since the sum of the integers

$$a_1 - a_2, a_2 - a_3, \ldots, a_{10} - a_1$$

is zero, it is not possible that only one of them is equal to ± 2 and the others are zero. Consequently, exactly two of them are equal to 1, and exactly two of them are equal to -1, and the remaining ones are equal to zero. Note that a_1 is equal to $-1, 0$ or 1.

Let $\mathcal{A}$ denote the set of solutions satisfying the given conditions, and $\mathcal{B}$ denote the set of 11-tuples of integers, whose first coordinate is at most 1 in absolute value, the remaining 10 coordinates add up to 0, and exactly two of these 10 coordinates are equal to 1, and exactly two of these 10 coordinates are equal to -1. Consider the map from $\mathcal{A} \to \mathcal{B}$, given by

$$(a_1, \ldots, a_{10}) \mapsto (a_1, a_1 - a_2, a_2 - a_3, \ldots, a_{10} - a_1).$$

Note that this map is a bijection. So the number of 10-tuples satisfying the given conditions is equal to

$$3\binom{10}{2}\binom{8}{2} = 3780.$$

■

Example 1.25 (India RMO 2013b P1)**.** Prove that there do not exist natural numbers x and y with $x > 1$ such that

$$\frac{x^7 - 1}{x - 1} = y^5 + 1.$$

Solution 21. Let x, y be natural numbers with $x \neq 1$ and satisfying the above equation. Note that the given equation can be rewritten as

$$y^5 = \frac{x^7 - x}{x - 1} = x(x^2 + x + 1)(x^3 + 1).$$

Observe that the integer x is coprime to $x^2 + x + 1$, and x is also coprime to $x^3 + 1$. Since

$$x^3 + 1 - (x - 1)(x^2 + x + 1) = 2$$

holds and the integer x^2+x+1 is odd, it follows that the integers x^2+x+1, x^3+1 are coprime. Since the positive integers $x, x^2 + x + 1, x^3 + 1$ are pairwise coprime, and their product is the fifth power of an integer, it follows that the integers $x, x^2 + x + 1, x^3 + 1$ are also fifth powers of positive integers. Using $1 = (x^3 + 1) - (x^3)$, we obtain that 1 can be expressed as the difference of the fifth powers of two distinct positive integers. However, this is impossible since the fifth powers of two distinct positive integers differ by at least 31. Indeed, if $i > j \geq 1$ are positive integers, then

$$\begin{aligned} i^5 - j^5 &= (i - j)(i^4 j + i^3 j + i^2 j^2 + ij^3 + j^4) \\ &\geq i^4 j + i^3 j + i^2 j^2 + ij^3 + j^4 \\ &\geq 2^4 + 2^3 + 2^2 + 2 + 1 \\ &= 31 \end{aligned}$$

holds. This completes the proof. ■

> **Remark.** It is worth comparing the above problem, and the following one.

Example 1.26 (IMOSL 2006 N5)**.** Find all integer solutions of the equation

$$\frac{x^7 - 1}{x - 1} = y^5 - 1.$$

The following is due to the AoPS user TomciO.

Solution 22. Let us establish the following claim.

> **Claim** — Let p, q be primes such that q divides the integer
>
> $$\frac{x^p - 1}{x - 1}$$
>
> for some integer $x \neq 1$. Then $q \equiv 1 \bmod p$ or $p = q$ holds.

Proof of the Claim. If $x \equiv 1 \bmod q$, then

$$\frac{x^p - 1}{x - 1} = x^{p-1} + x^{p-2} + \cdots + 1 \equiv p \bmod q$$

holds, which shows that $p = q$.

If $x \not\equiv 1 \bmod q$, then let k denote the smallest positive integer such that $x^k \equiv 1 \bmod q$ holds. Writing $p = ak + b$ for some integers a, b with $0 \leq b < k$, and using the congruence $x^p \equiv 1 \bmod q$ combined with the minimality of k, it follows that $b = 0$, and hence p is a multiple of the integer k. Since k is larger than 1 and p is a prime, we obtain $p = k$. A similar argument also shows that k divides $q - 1$, and consequently, p divides $q - 1$. □

Let x, y be integers with $x \neq 1$ and satisfying the given equation.

Let us first consider the case that $x \equiv 1 \bmod 7$. We obtain

$$y^5 - 1 = x^6 + x^5 + \cdots + x + 1 \equiv 0 \bmod 7,$$

which yields $y \equiv 1 \bmod 7$. This shows that

$$y^4 + y^3 + y^2 + y + 1 \equiv 5 \bmod 7,$$

and hence, the integer $y^4 + y^3 + y^2 + y + 1$ admits a prime divisor q satisfying $q \not\equiv 1 \bmod 7$ and $q \neq 7$. Note that q divides $(x^7 - 1)/(x - 1)$, and applying the Claim, we obtain a contradiction.

Let us now consider the case that $x \not\equiv 1 \bmod 7$. Note that

$$(x-1)\left(\frac{x^7-1}{x-1} - 1\right) = x^7 - 1 - (x - 1) \equiv 0 \bmod 7$$

holds, and using $x - 1 \not\equiv 0 \bmod 7$, we obtain

$$\frac{x^7-1}{x-1} \equiv 1 \bmod 7,$$

which yields $y^5 \equiv 2 \bmod 7$. This gives that

$$\begin{aligned} y &\equiv y^{25} \bmod 7 \\ &\equiv 2^5 \bmod 7 \\ &\equiv 4 \bmod 7, \end{aligned}$$

and hence $y - 1 \equiv 3 \bmod 7$. It follows that some prime divisor q of $y - 1$ satisfies $q \not\equiv 1 \bmod 7$ and $q \neq 7$. Note that q divides $(x^7 - 1)/(x - 1)$, and applying the Claim, we obtain a contradiction.

We conclude that there are no integer solutions to the given equation satisfying $x \neq 1$. ■

Example 1.27 (India RMO 2013c P2)**.** Find all 4-tuples (a, b, c, d) of natural numbers with $a \leq b \leq c$ and $a! + b! + c! = 3^d$.

Solution 23. Let a, b, c, d be natural numbers satisfying the given conditions. Note that $d \geq 1$. If $a \geq 2$, then $3^d = a! + b! + c!$ would be even, which is impossible. So a is equal to 1, which gives $b! + c! = 3^d - 1$. This implies that $b \leq 2$. If $b = 1$, then $c! = 3^d - 2$, which shows $c \leq 2$, and hence $c = 1, d = 1$. Suppose b is equal to 2. Then we obtain $c! = 3^d - 3$, which gives $d \geq 2$. Since 9 does not divide $3^d - 3$, it follows that $c \leq 5$. This implies that (c, d) is equal to $(3, 2)$ or $(4, 3)$. This gives that (a, b, c, d) is equal to one of

$$(1, 1, 1, 1), (1, 2, 3, 2), (1, 2, 4, 3).$$

Note that the above tuples also satisfy the given conditions. This shows that the tuples are precisely the 4-tuples of natural numbers satisfying the given conditions. ∎

Example 1.28 (India RMO 2014c P6)**.** For any natural number, let $S(n)$ denote sum of digits of n. Find the number of 3 digit numbers for which $S(S(n)) = 2$.

Solution 24. Let n be a 3-digit natural number such that $S(S(n)) = 2$. Note that $S(n) \leq 27$. Using $S(n) \leq 27$ and $S(S(n)) = 2$, it follows that $S(n)$ is equal to one of $2, 11, 20$. Noting that $S(n)$ is congruent to 2 mod 9, and using $n \equiv S(n) \bmod 9$, we obtain $n \equiv 2 \bmod 9$. Conversely, note that if m is a 3-digit number and $m \equiv 2 \bmod 9$, then $S(m) \equiv 2 \bmod 9$ holds, and using $S(m) \leq 27$, it follows that $S(m)$ is equal to one of $2, 11, 20$. We conclude that the 3-digit numbers n satisfying $S(S(n)) = 2$ are precisely the 3-digit numbers which are congruent to 2 modulo 9, or equivalently, n is equal to one of

$$101, 110, \dots, 992.$$

Hence there are

$$1 + \frac{1}{9}(992 - 101) = 100$$

three-digit numbers n for which $S(S(n)) = 2$. ∎

Example 1.29 (India RMO 2016a P3)**.** For any natural number n, expressed in base 10, let $S(n)$ denote the sum of all digits of n. Find all natural numbers n such that $n = 2S(n)^2$.

Solution 25. Let n be a natural number satisfying the given conditions. Using $n \equiv S(n) \bmod 9$, it follows that

$$n \equiv 2n^2 \bmod 9,$$

which gives

$$2n^2 \equiv n \bmod 3,$$

which implies that n is congruent to one of $0, 2$ modulo 3. Consequently, n is congruent to one of $0, 3, 6, 2, 5, 8$ modulo 9. Using $n \equiv 2n^2 \bmod 9$, we obtain that n is congruent to one of $0, 5$ modulo 9.

Note that if n has d digits, then

$$10^{d-1} \leq n = 2S(n)^2 \leq 2(9d)^2$$

holds. Note that $10^{5-1} > 2(9 \cdot 5)^2$ holds, and if k is a positive integer such that $10^{k-1} > 2(9k)^2$ holds, then we obtain

$$\begin{aligned} \frac{10^k}{2(9(k+1))^2} &> \frac{10 \times 2(9k)^2}{2(9(k+1))^2} \\ &= \frac{10}{(1+1/k)^2} \\ &\geq \frac{10}{2^2} \\ &> 1. \end{aligned}$$

It follows that $10^{k-1} > 2(9k)^2$ holds for any integer $k \geq 5$. We obtain that $d \leq 4$, that is, n has at most four digits. This gives that $S(n) \leq 36$. Since $S(n) \equiv n \bmod 9$ and n is congruent to one of $0, 5$ modulo 9, it suffices to consider the following cases.

Let us consider the case that $S(n)$ is a multiple of 9. Then $S(n)$ is equal to one of $9, 18, 27, 36$, and n is equal to one of $162, 648, 1458, 2592$. Note that none of the integers $1458, 2592$ satisfies the given condition. This shows that n is one of $162, 648$.

Let us now consider the case that $S(n)$ is equal to one of $5, 14, 23, 32$. Then $S(n)$ is equal to one of $50, 392, 1058, 2048$. Note that none of $1058, 2048$ satisfies the given condition, which implies that n is equal to one of $50, 392$.

It follows that n is equal to one of the integers

$$50, 162, 392, 648.$$

Note that the above integers satisfy the given condition. Hence, the above are precisely all the natural numbers satisfying $n = S(n)^2$. ■

Example 1.30 (India RMO 2016b P3)**.** For any natural number n, expressed in base 10, let $S(n)$ denote the sum of all digits of n. Find all natural numbers n such that $n^3 = 8S(n)^3 + 6nS(n) + 1$.

Solution 26. Let n be a natural number satisfying the given condition. Using $S(n) \equiv n \bmod 9$, we obtain

$$7n^3 + 6n^2 + 1 \equiv 0 \bmod 9.$$

Note that

$$\begin{aligned}7n^3 + 6n^2 + 1 &= (n+1)(7n^2 - n + 1)\\ &\equiv (n+1)(n+1)(7n+1) \bmod 9\\ &\equiv 7(n+1)(n+1)(n+4) \bmod 9\end{aligned}$$

holds, which implies that 3 divides $n+1$ or 9 divides $n+4$.

Note that

$$8S(n)^3 + 6nS(n) + 1 - n^3 = (2S(n))^3 + (-n)^3 + 1^3 - 3(2S(n))(-n)$$

holds. This shows that

$$2S(n) + 1 = n,$$

or

$$2S(n) = 1 = -n$$

holds, and hence, we get

$$2S(n) + 1 = n.$$

Suppose n has d digits. Note that

$$\begin{aligned}10^{d-1} &\leq n\\ &= 2S(n) + 1\\ &\leq 2 \cdot 9d + 1\end{aligned}$$

holds. Observe that $10^{3-1} > 18 \cdot 3 + 1$ holds, and if k is a positive integer satisfying $10^{k-1} > 18k + 1$, then

$$\begin{aligned}\frac{10^k}{18k+19} &> \frac{10(18k+1)}{18k+19}\\ &> 1\end{aligned}$$

holds. By induction, it follows that $d \leq 2$, that is, n has at most two digits. It follows that $S(n) \leq 18$.

Note that $n = 2S(n) + 1$ implies that n is odd. Moreover, 3 divides $n+1$ or 9 divides $n+4$. Combining these with the congruence $S(n) \equiv n \bmod 9$, we obtain that $S(n)$ is equal to one of $5, 11, 17$. Note that none of the integers $5, 11$ satisfies $n = 2S(n) + 1$. This shows that $n = 17$. Since $n = 17$ satisfies $2S(n) + 1 = n$, it also satisfies the given condition. It follows that $n = 17$ is the only natural number satisfying the given condition. ■

Example 1.31 (India RMO 2023b P3)**.** For any natural number n, expressed in base 10, let $s(n)$ denote the sum of all its digits. Find all natural numbers m and n such that $m < n$ and

$$(s(n))^2 = m \text{ and } (s(m))^2 = n.$$

Solution 27. Let m, n be natural numbers satisfying the given conditions. Note that

$$n^2 \equiv m \bmod 9, m^2 \equiv n \bmod 9,$$

which implies that m is congruent to one of $0, 1$ modulo 3, and so is n, and consequently, it follows that

$$m \equiv n \bmod 3.$$

Suppose m has d digits. Note that $m \geq 10^{d-1}$ and $s(m) \leq 9d$ holds. Consequently, $10^{d-1} > (9d)^2$ would yield

$$n > m \geq 10^{d-1} > (9d)^2 \geq s(m)^2 = n,$$

which is impossible. Observe that $10^{5-1} > (9 \cdot 5)^2$ holds, and if k is a positive integer satisfying $10^{k-1} > (9k)^2$, then

$$\begin{aligned}\frac{10^k}{(9(k+1))^2} &> \frac{10(9k)^2}{(9(k+1))^2} \\ &= \frac{10}{(1+\frac{1}{k})^2} \\ &> 1\end{aligned}$$

holds. By induction, it follows that $d \leq 4$, that is, m has at most 4 digits.

This gives

$$s(m) \leq 9d \leq 36,$$

and hence, it follows that

$$n = s(m)^2 \leq 36^2 = 1296.$$

This yields

$$s(n) \leq \max\{s(1296), s(1199), s(1099), s(999)\} = \max\{18, 20, 19, 27\} = 27,$$

which gives

$$m = s(n)^2 \leq 27^2 = 729.$$

We get

$$s(m) \leq \max\{s(729), s(719), s(709), s(699)\} = \max\{18, 17, 16, 24\} = 24,$$

which shows that

$$n = s(m)^2 \leq 24^2 = 576.$$

This gives

$$s(n) \leq \max\{s(576), s(569), s(559), \dots, s(519), s(509), s(499)\} = 22,$$

which yields
$$m = s(n)^2 \leq 22^2 = 484.$$
This shows that
$$s(m) \leq \max\{s(484), s(479), s(399)\} = 21,$$
which gives
$$n = s(m)^2 \leq 21^2 = 441.$$
It follows that
$$s(n) \leq \max\{s(441), s(400), s(361), s(324), s(289)\} = \max\{9, 4, 10, 9, 19\} = 19,$$
and hence, we get
$$m = s(n)^2 \leq 19^2 = 361.$$
This gives
$$s(m) \leq \max\{s(361), s(324), s(289)\} = 19,$$
which yields
$$n \leq 19^2,$$
and using the inequality $m < n$, the congruence $m \equiv n \bmod 3$, and that m is a perfect square, we obtain
$$m \leq 17^2.$$
Note that
$$s(17^2)^2 = s(289)^2 = 19^2 = 361, \quad s(361)^2 = 100 \neq 17^2$$
holds, which implies that
$$m \leq 16^2,$$
and hence, we obtain
$$s(m) \leq \max\{s(256), s(225), s(196), s(169), s(81), s(64), s(49)\} = 16.$$
Consequently, we obtain
$$n \leq 16^2,$$
and using the inequality $m < n$, the congruence $m \equiv n \bmod 3$, and that m is a perfect square, we get
$$m \leq 14^2.$$
Observe that
$$s(14^2)^2 = s(196)^2 = 16^2 = 256, \quad s(256)^2 = 13^2 \neq 14^2$$
holds, which yields
$$m \leq 13^2.$$

Note that $(m, n) = (13^2, 16^2)$ is a solution to the given equation. If $8^2 \leq m \leq 12^2$, then

$$s(m) \leq \max\{s(64), s(81), s(100), s(121), s(144)\} = 9,$$

which gives $n \leq 81$, and hence (m, n) is equal to $(8^2, 9^2)$, which is impossible. It follows that $m \leq 7^2$. Observe that $s(m)^2 = n, s(n)^2 = m$ fails to hold if m is any of $1^2, 2^2, 3^2, 4^2, 5^2, 6^2, 7^2$.

We conclude that $(m, n) = (13^2, 16^2)$ is the only solution to the given equations. ■

Remark. The above solution to Example 1.31 makes an effort to defer computing the (decimal expansions of the) perfect squares. It turns out that without deferring it, we may arrive at the solution in fewer steps.

After obtaining $n \leq 36^2 = 1296$, one may compute the sum of the digits of the perfect squares lying between 11^2 and 36^2, as recorded in the table below.

k	k^2	$s(k^2)$
36	1296	18
35	1225	10
34	1156	13
33	1089	18
32	1024	7
31	961	16
30	900	9
29	841	13
28	784	19
27	729	18
26	676	19
25	625	13
24	576	18
23	529	16
22	484	16
21	441	9
20	400	4
19	361	10
18	324	9
17	289	19
16	256	13
15	225	9
14	196	16
13	169	16
12	144	9
11	121	4

This yields
$$s(n) \leq 19,$$
which gives
$$m = s(n)^2 \leq 19^2 = 361.$$
We get
$$s(m) \leq 19,$$
which shows that
$$n = s(m)^2 \leq 19^2 = 361.$$
Since $m \equiv n \bmod 3$, we obtain
$$m \leq 17^2 = 289.$$
Note that
$$s(17^2)^2 = s(289)^2 = 19^2 = 361, \quad s(361)^2 = 100 \neq 17^2$$
holds, which implies that
$$m \leq 16^2,$$
and hence, we obtain
$$s(m) \leq \max\{s(256), s(225), s(196), s(169), s(81), s(64), s(49)\} = 16.$$
Consequently, we obtain
$$n \leq 16^2,$$
and this gives
$$m \leq 15^2.$$
Observe that
$$s(15^2)^2 = s(225)^2 = 81, \quad s(81)^2 = 81 \neq 15^2,$$
$$s(14^2)^2 = s(196)^2 = 16^2 = 256, \quad s(256)^2 = 13^2 \neq 14^2$$
holds, which yields
$$m \leq 13^2.$$
Note that $(m, n) = (13^2, 16^2)$ is a solution to the given equation. If $8^2 \leq m \leq 12^2$, then
$$s(m) \leq \max\{s(64), s(81), s(100), s(121), s(144)\} = 9,$$
which gives $n \leq 81$, and hence (m, n) is equal to $(8^2, 9^2)$, which is impossible. It follows that $m \leq 7^2$. Observe that $s(m)^2 = n, s(n)^2 = m$ fails to hold if m is any of $1^2, 2^2, 3^2, 4^2, 5^2, 6^2, 7^2$.

We conclude that $(m, n) = (13^2, 16^2)$ is the only solution to the given equations.

Example 1.32 (India RMO 2014e P2)**.** The roots of the equation
$$x^3 - 3ax^2 + bx + 18c = 0$$

form a non-constant arithmetic progression and the roots of the equation

$$x^3 + bx^2 + x - c^3 = 0$$

form a non-constant geometric progression. Given that a, b, c are real numbers, find all positive integral values a and b.

Solution 28. Let a, b, c are real numbers such that the given conditions hold. Let $d \neq 0$ denote the common difference of the non-constant arithmetic progression formed by the roots of $x^3 - 3ax^2 + bx + 18c = 0$, and let r denote the common ratio of the non-constant geometric progression formed by the roots of $x^3 + bx^2 + x - c^3 = 0$. Note that the terms of the arithmetic progression are $a - d, a, a + d$, and the terms of the geometric progression are $\frac{c}{r}, c, cr$. It follows that

$$a(a-d) + a(a+d) + (a^2 - d^2) = b, \quad a(a^2 - d^2) = -18c,$$

which gives

$$d^2 = 3a^2 - b, \quad a(a^2 - d^2) = -18c,$$

and

$$\frac{c}{r} + c + cr = -b, \frac{c^2}{r} + c^2 + c^2 r = 1.$$

Eliminating d, we obtain

$$a(4a^2 - b) = -18c,$$

and eliminating r, we obtain

$$bc = -1.$$

Eliminating c yields

$$ab(b - 2a^2) = 18. \tag{7}$$

Assume that a, b are positive integers. Since a, b are positive, it follows that $b - 2a^2$ is also positive. Note that $b, b - 2a^2$ are of the same parity. Since 2 divides 18 and 4 does not divide 18, we obtain $a = 2$, and hence

$$b(b-8) = 9.$$

Since b is positive, it gives $b = 9$.

Note that for $(a, b) = (2, 9)$, and for $c = -\frac{1}{9}$, the roots of the polynomial $x^3 - 3ax^2 + bx + 18c$ are $2 \mp \sqrt{3}, 2, 2 \pm \sqrt{3}$, which form a non-constant arithmetic progression, and the roots of the polynomial $x^3 + bx^2 + x - c^3$ are $c/r, c, cr$ with $r = 40 \pm \sqrt{40^2 - 1}$, which form a non-constant geometric progression.

This proves that the required solution for (a, b) is precisely $(2, 9)$. ■

Example 1.33 (India RMO 2015d P3)**.** Find all integers a, b, c such that $a^2 = bc + 1$ and $b^2 = ca + 1$.

Solution 29. Let a, b, c be integers satisfying the given equations.

Let us first consider the case that $a = b$. Note that $a(a-c) = 1$ holds, which shows that $(a, a-c)$ is equal to one of $(1,1), (-1,-1)$, which implies that (a, b, c) is equal to one of $(1,1,0), (-1,-1,0)$.

Now, let us consider the case that $a \neq b$. Taking the difference of the given equations, we obtain $(a-b)(a+b+c) = 0$, which gives $a+b = -c$. Substituting $c = -a-b$ in $a^2 = bc+1$ yields $a^2 + ab + b^2 = 1$, which is equivalent to

$$(2a+b)^2 + 3b^2 = 4.$$

This shows that $(2a+b, b)$ is equal to one of

$$(1,1), (1,-1), (-1,1), (-1,-1), (2,0), (-2,0).$$

This implies that (a, b, c) is equal to one of

$$(0,1,-1), (1,-1,0), (-1,1,0), (0,-1,1), (1,0,-1), (-1,0,1).$$

Combining the above cases, it follows that (a, b, c) is equal to one of

$$(1,1,0), (-1,-1,0), (0,1,-1), (1,-1,0), (-1,1,0), (0,-1,1), (1,0,-1), (-1,0,1).$$

Note that any of the above triples satisfies the given equations. Hence, the solutions of the given equations over the integers are precisely the above eight triples. ■

Example 1.34 (India RMO 2015b P3)**.** Find all integers a, b, c such that $a^2 = bc + 4$ and $b^2 = ca + 4$.

Solution 30. Let a, b, c be integers satisfying the given equations.

Let us first consider the case that $a = b$. Note that $a(a-c) = 4$ holds, which shows that $(a, a-c)$ is equal to one of the elements of

$$\{(d, d - 4/d) \,|\, d \text{ is a divisor of } 4\},$$

and hence (a, b, c) is equal to one of

$$(1,1,-3), (-1,-1,3), (2,2,0), (-2,-2,0), (4,4,3), (-4,-4,-3).$$

Now, let us consider the case that $a \neq b$. Note that

$$a^3 - b^3 = a(bc+4) - b(ca+4) = 4(a-b)$$

holds, which yields

$$a^2 + ab + b^2 = 4.$$

It follows that at least one of a, b is even, and hence, both of them are even. Observe that

$$\left(a + \frac{b}{2}\right)^2 + 3\left(\frac{b}{2}\right)^2 = 4$$

holds, which shows that $(a+b/2, b/2)$ is equal to one of

$$(2,0),(-2,0),(1,1),(1,-1),(-1,1),(-1,-1),$$

and hence (a,b) is equal to one of

$$(2,0),(-2,0),(0,2),(2,-2),(-2,2),(0,-2).$$

This implies that (a,b,c) is equal to one of

$$(2,0,-2),(-2,0,2),(0,2,-2),(2,-2,0),(-2,2,0),(0,-2,2).$$

Considering the above cases, it follows that (a,b,c) is equal to one of

$$(1,1,-3),(-1,-1,3),(2,2,0),(-2,-2,0),(4,4,3),(-4,-4,-3),$$

$$(2,0,-2),(-2,0,2),(0,2,-2),(2,-2,0),(-2,2,0),(0,-2,2).$$

Note that any of the above pairs satisfy the given equations. This proves that the above tuples are precisely all the solutions of the given equation over the integers. ■

Example 1.35 (India RMO 2015c P4)**.** Find all three digit natural numbers of the form $(abc)_{10}$ such that $(abc)_{10}$, $(bca)_{10}$, $(cab)_{10}$ are in geometric progression. (Here $(abc)_{10}$ is representation in base 10).

Solution 31. Let $(abc)_{10}$ be a three digit natural number such that $(abc)_{10}$, $(bca)_{10}$, $(cab)_{10}$ are in geometric progression, that is,

$$(100b+10c+a)^2 = (100a+10b+c)(100c+10a+b)$$

holds, which is equivalent to

$$\begin{aligned}&10000b^2+100c^2+a^2+2000bc+200ab+20ca\\&=10000ca+1000(a^2+bc)+100(c^2+2ab)+10(b^2+ca)+bc.\end{aligned}$$

This implies

$$10000(b^2-ca)-1000(a^2-bc)-10(b^2-ca)+(a^2-bc)=0,$$

which reduces to

$$(10(b^2-ca)-(a^2-bc))(1000-1)=0.$$

This gives

$$(10a-b)c = 10b^2-a^2. \tag{8}$$

Note that $10a-b$ is nonzero, otherwise, 10 would divide b, implying $b=0$, and thus, $a=0$, which is impossible. It follows that

$$\begin{aligned} c &= \frac{10b^2-a^2}{10a-b} \\ &= \frac{10b(b-10a)+100ab-a^2}{10a-b} \\ &= -10b+\frac{100ab-a^2}{10a-b} \\ &= -10b+a\frac{100b-a}{10a-b} \\ &= -10b+a\frac{100(b-10a)+1000a-a}{10a-b} \\ &= -10b-100a+\frac{999a^2}{10a-b}. \end{aligned}$$

Let d denote the greatest common divisor of a, b. Note that $10a-b$ divides $999a^2$, which shows that $10\frac{a}{d}-\frac{b}{d}$ divides $999d\left(\frac{a}{d}\right)^2$. Since $a/d, b/d$ are relatively prime, it follows that $10\frac{a}{d}-\frac{b}{d}$ divides $999d$. Using Eq. (8), it follows that $b \geq 1$. Note that if $a=b$ holds, then using Eq. (8), we obtain $a=b=c$, and hence, $(abc)_{10}$, $(bca)_{10}$, $(cab)_{10}$ are equal, and form a geometric progression. It remains to consider the case that $a \neq b$, which we assume from now on. Since $1 \leq b \leq 9$ and d divides b, it follows that $d \leq 4$.

Let us consider the case that $d=4$. Then $a/d, b/d$ belong to $\{1,2\}$. Using $a \neq b$, it follows that $\left(\frac{a}{d}, \frac{b}{d}\right)$ is equal to $(1,2)$ or $(2,1)$, which implies that $10\frac{a}{d}-\frac{b}{d}$ does not divide $999d$, which is impossible. We obtain $d \neq 4$.

Let us consider the case that $d=3$. Note that $1 \leq a/d \leq 3$, $1 \leq b/d \leq 3$ hold, and hence $7 = 10-3 \leq 10\frac{a}{d}-\frac{b}{d} \leq 30-1 = 29 < 37$. Since $10\frac{a}{d}-\frac{b}{d}$ divides $999d = 37 \times 81$, it is equal to one of $9, 27$, which shows that $\frac{a}{d}-\frac{b}{d} \equiv 0 \bmod 3$. This yields $a=b$, which is impossible. We get that $d \neq 3$.

Let us consider the case that $d=2$. Note that $1 \leq a/d \leq 4, 1 \leq b/d \leq 4$ hold, which yields $6 \leq 10a/d - b/d \leq 39$. Since $10a/d-b/d$ divides $999d = 2 \times 27 \times 37$, it is equal to one of $27, 37$, and it follows that $(a/d, b/d)$ is equal to one of $(3,3), (4,3)$. Using $a \neq b$, we get that (a,b) is equal to $(8,6)$. Then Eq. (8) yields $c=4$.

Let us consider the case that $d=1$. Note that $10a-b$ divides $999 = 27 \times 37$. Observing that $10a-b \leq 90$, we obtain $10a-b$ is equal to one of $1, 3, 9, 27, 37$, and hence (a,b) is equal to one of $(1,9), (1,7), (1,1), (3,3), (4,3)$. Note that (a,b) is equal to none of $(1,9), (1,7)$, otherwise, we would get $c > 10$. Using $a \neq b$, it follows that (a,b) is equal to $(4,3)$. Then Eq. (8) yields $c=2$.

The above argument shows that $(abc)_{10}$ is equal to one of the following eleven integers

$$111, 222, 333, 444, 555, 666, 777, 888, 999, 432, 864.$$

Note that

$$\begin{aligned} 432 \times 243 &= 144 \times 729 \\ &= 12^2 \times 27^2 \\ &= 324^2, \\ 864 \times 486 &= 4 \times 432 \times 243 \\ &= 4 \times 324^2 \\ &= 648^2 \end{aligned}$$

holds. This proves that the above eleven integers satisfy the required condition, and hence, these are precisely all the required three digit natural numbers. ■

Example 1.36 (India RMO 2016f P1)**.** Find distinct positive integers $n_1 < n_2 < \cdots < n_7$ with the least possible sum, such that their product $n_1 \times n_2 \times \cdots \times n_7$ is divisible by 2016.

Solution 32. Note that

$$2016 = 32 \times 63 = 2^5 \cdot 3^2 \cdot 7$$

holds.

Let $n_1 < n_2 < \cdots < n_7$ be distinct positive integers satisfying the given conditions. Note that $n_i \geq i$ holds for all $1 \leq i \leq 7$, which gives

$$n_1 + n_2 + \cdots + n_7 \geq 1 + 2 + 3 + 4 + 5 + 6 + 7.$$

Note that the product of the distinct integers $1, 2, 3, 4, 6, 7, 8$ is equal to $2^5 \cdot 3^2 \cdot 7$. This implies that

$$n_1 + n_2 + \cdots + n_7 \leq 3 + 1 + 2 + 3 + 4 + 5 + 6 + 7. \tag{9}$$

Note that $n_4 > 4$ implies that

$$n_1 + n_2 + \cdots + n_7 \geq 1 + 2 + 3 + 5 + 6 + 7 + 8 = 4 + 1 + 2 + 3 + 4 + 5 + 6 + 7,$$

which is impossible. This gives that $n_4 \leq 4$, and consequently,

$$n_1 = 1, n_2 = 2, n_3 = 3, n_4 = 4$$

holds.

If

$$n_1 + n_2 + \cdots + n_7 = 1 + 2 + 3 + 4 + 5 + 6 + 7$$

holds, then n_i is equal to i for any $1 \leq i \leq 7$, and this is impossible since $7!$ is not divisible by the divisor 2^5 of 2016. This proves that

$$n_1 + n_2 + \cdots + n_7 > 1 + 2 + 3 + 4 + 5 + 6 + 7,$$

which implies that $n_i > i$ for some $1 \leq i \leq 7$, and hence $n_i > i$ for some $i \geq 5$. This shows that

$$n_1 + n_2 + \cdots + n_7 \geq 1 + 2 + 3 + 4 + 6 + 7 + 8 = 3 + 1 + 2 + 3 + 4 + 5 + 6 + 7.$$

Using Eq. (9), we obtain

$$n_1 = 1, n_2 = 2, n_3 = 3, n_4 = 4, n_5 = 6, n_6 = 7, n_7 = 8.$$

■

§1.6 Using congruences

Example 1.37 (Jonquières 1878)**.** Prove that the equation

$$y^2 = x^3 + 23$$

has no solutions in integers.

Solution 33. Let x, y be integers satisfying the given equation. If x is even, then y is odd, and hence $y^2 \equiv 1 \bmod 8$, which is impossible since $x^3 + 23 \equiv 7 \bmod 8$. It follows that x is odd, and consequently, y is even. Write $y = 2k$ for some integer k. Note that

$$4k^2 + 4 = x^3 + 3^3$$

holds, which yields

$$(x + 3)(x^2 - 3x + 9) = 4(k^2 + 1).$$

Since x is odd, it follows that $x^2 - 3x + 9$ is odd, and hence 4 divides $x + 3$, or equivalently, we have $x \equiv 1 \bmod 4$. This shows that

$$x^2 - 3x + 9 \equiv 3 \bmod 4.$$

It follows that $x^2 - 3x + 9$ admits a prime factor p satisfying $p \equiv 3 \bmod 4$. This implies that p divides $k^2 + 1$, and hence the order of k modulo p is 4. Applying division algorithm, and using $k^{p-1} \equiv 1 \bmod 4$, we obtain that 4 divides $p - 1$, which is impossible. This proves that the given equation has no solution over the integers. ■

> **Remark.** We refer to this notes by Keith Conrad on Mordell's equation, which are equations of the form $y^2 = x^3 + k$, where k is an integer.

Example 1.38 (India RMO 1995 P4)**.** Show that the quadratic equation $x^2 + 7x - 14(q^2 + 1) = 0$, where q is an integer, has no integer root.

 The content posted here and at this blog by Evan Chen are quite useful.

Solution 34. It suffices to show that the discriminant of the polynomial $x^2 + 7x - 14(q^2 + 1)$ is not a perfect square, that is, the integer $49 + 56(q^2 + 1)$ is not perfect square.

Let us assume that $49 + 56(q^2 + 1)$ is perfect square. Note that it is divisible by 7. So it is divisible by 49. Hence, $q^2 + 1$ is divisible by 7, which is impossible since $7 \equiv 3 \bmod 4$. Alternatively, note that

$$q^2 + 1 \equiv 1, 2, 5, 3 \bmod 7$$

if $q \equiv 0, \pm 1, \pm 2, \pm 3 \bmod 7$, and thus $q^2 + 1$ is not divisible by 7. This shows that the discriminant of $x^2 + 7x - 14(q^2 + 1)$ is not a perfect square. So this polynomial has no integer root. ■

Example 1.39 (India RMO 2009 P2)**.** Show that there is no integer a such that $a^2 - 3a - 19$ is divisible by 289.

Solution 35. If $a^2 - 3a - 19$ is divisible by 289 for some integer a, then

$$4(a^2 - 3a - 19) = (2a)^2 - 12a - 76 = (2a - 3)^2 - 85$$

would be divisible by $289 = 17^2$, and hence $2a - 3$ would be a multiple of 17, and consequently, 17^2 would divide 85, which is impossible. This proves the result. ■

Example 1.40 (Mathematical Ashes 2011 P2)**.** Find all pairs (m, n) of nonnegative integers for which

$$m^2 + 2 \cdot 3^n = m(2^{n+1} - 1).$$

Walkthrough —

(a) Let m, n be nonnegative integers satisfying the given equation. Considering the roots of $x^2 - x(2^{n+1} - 1) + 2 \cdot 3^n$, it follows that

$$3^k + 2 \cdot 3^\ell = 2^{n+1} - 1$$

holds, for some nonnegative integers k, ℓ satisfying $k + \ell = n$.

(b) Show that if $n \geq 6$, then $\min\{k, \ell\} \geq 2$ holds. Note that

$$3^k < 2^{n+1} < 9^{(n+1)/3}$$

holds, implying $k < 2(n+1)/3$. Also note that

$$2 \cdot 3^\ell < 2^{n+1} < 2 \cdot 3^{2n/3}$$

holds, implying $\ell < 2n/3$. Using $k + \ell = n$, it follows that

$$k > \frac{n-2}{3}, \ell > \frac{n-2}{3}.$$

(c) Let us consider the case[a] that $n \geq 6$. Note that $m := \min\{k, \ell\} \geq 2$ holds.

(i) Note that 9 divides $2^{n+1} - 1$, and show that 6 divides $n+1$. Writing $n + 1 = 6j$ yields

$$2^{n+1} - 1 = (4^j - 1)(4^{2j} + 4^j + 1) = (2^j - 1)(2^j + 1)((4^j - 1)^2 + 3 \cdot 4^j).$$

(ii) Noting that $(4^j - 1)^2 + 3 \cdot 4^j$ is divisible by 3, but not by 9, and that the integers $2^j - 1, 2^j + 1$ are coprime, conclude that 3^{m-1} divides one of $2^j - 1, 2^j + 1$.

(iii) Prove that

$$3^{m-1} \leq 2^j + 1 \leq 3^j = 3^{\frac{n+1}{6}},$$

implying

$$m - 1 \leq \frac{n+1}{6}.$$

(iv) Conclude that

$$\frac{n-2}{3} - 1 < m - 1 \leq \frac{n+1}{6}.$$

holds.

(v) This yields $n < 11$, contradicting $n \geq 6$ and 6 divides $n + 1$.

(d) It remains to consider the case $n \leq 5$.

[a] It also suffices to assume that $n \geq 5$ holds to obtain $m \geq 2$.

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