Quadratic polynomials

MOPSS

26 April 2025

Mathematics Olympiad

Problem Solving Sessions

MOPSS

Department of Mathematics
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https://jpsaha.github.io/MOTP/MOPSS/

Suggested readings

- Evan Chen's advice On reading solutions, available at https://blog.evanchen.cc/2017/03/06/on-reading-solutions/.
- Evan Chen's Advice for writing proofs/Remarks on English, available at https://web.evanchen.cc/handouts/english/english.pdf.
- Notes on proofs by Evan Chen from OTIS Excerpts [**Che25**, Chapter 1].
- Tips for writing up solutions by Edward Barbeau, available at https://www.math.utoronto.ca/barbeau/writingup.pdf.
- Evan Chen discusses why math olympiads are a valuable experience for high schoolers in the post on Lessons from math olympiads, available at https://blog.evanchen.cc/2018/01/05/lessons-from-math-olympiads/.

List of problems and examples

§1 Quadratic polynomials

Example 1.1 (India RMO 1991 P6)**.** Find all integer values of a such that the quadratic expression

$$(x+a)(x+1991)+1$$

can be factored as a product $(x+b)(x+c)$ where b and c are integers.

Solution 1. Let us establish the following claim.

Claim — A monic quadratic polynomial $f(x)$ with integer coefficients factorizes into the product of two monic polynomials with integer coefficients if and only if the discriminant of $f(x)$ is a perfect square.

Proof of the Claim. Since the discriminant of $f(x)$ is equal to the square of the twice of the difference of its roots, the "only if part" follows.

To prove the "if part", assume that the discriminant of $f(x)$ is a perfect square. Write $f(x) = x^2 + Bx + C$. Note that $B^2 - 4C = n^2$ for some integer n. This shows that the integers B, n are of the same parity. It follows that

$$-B+n, -B-n$$

are even integers. This implies that the polynomial $f(x)$ have integer roots, proving the "if part" of the Claim. □

By the above Claim, the given problem is equivalent to determining all the integers a such that the discriminant of

$$(x+a)(x+1991)+1$$

is a perfect square, that is,

$$(a+1991)^2 - 4(1991a+1) = n^2$$

holds for some integer n.

Let a, n be integers satisfying the above condition. Note that the above can be rewritten as

$$(a-1991)^2 - n^2 = 4,$$

which is equivalent to

$$(a - 1991 - n)(a - 1991 + n) = 4.$$

Noting that the integers $a - 1991 - n, a - 1991 + n$ are of the same parity, it follows that the above condition is equivalent to $(a - 1991 - n, a - 1991 + n)$ being equal to one of $(2, 2), (-2, -2)$, which holds if and only if (a, n) is equal to one of $(1993, 0), (1989, 0)$.

This proves that the integer values of a satisfying the required condition are precisely $1989, 1993$. ■

Example 1.2 (India RMO 1999 P7)**.** Find the number of quadratic polynomials, $ax^2 + bx + c$, which satisfy the following conditions:

1. a, b, c are distinct,
2. $a, b, c \in \{1, 2, 3, \ldots, 1999\}$ and
3. $x + 1$ divides $ax^2 + bx + c$.

Solution 2. Note the third condition is equivalent to $a + c = b$. So the given problem is equivalent to finding the number of triples (a, b, c) where a, b, c are distinct integers lying between 1 and 1999, and satisfy $a + c = b$. Note that it is equal to the number of pairs (a, c) of distinct integers a, c lying between 1 and 1999, such that their sum $a + c$ is at most 1999. Observe that these are precisely the pairs of the form (a, c) where a, c are integers satisfying $1 \leq a \leq 1998$, $1 \leq c \leq 1999 - a$ and $a \neq c$. Note that the number of such pairs is equal to

$$\sum_{i=1}^{1998} (1999 - i) - 999,$$

where the sum accounts for the pairs (a, c) satisfying $1 \leq a \leq 1998$ and $1 \leq c \leq 1999 - a$, and the term 999 accounts for the pairs (a, c) satisfying these two conditions and the condition $a = c$. We conclude that the number of the quadratic polynomials, satisfying the given conditions, is equal to

$$\begin{aligned}
\sum_{i=1}^{1998} (1999 - i) - 999 &= \sum_{i=1}^{1998} i - 999 \\
&= 999 \times 1999 - 999 \\
&= 1998000 - 1998 \\
&= 1996002.
\end{aligned}$$

■

Example 1.3 (India RMO 2013a P6)**.** Suppose that m and n are integers such that both the quadratic equations $x^2 + mx - n = 0$ and $x^2 - mx + n = 0$ have integer roots. Prove that n is divisible by 6.

Solution 3. Since the given quadratic polynomials have integer roots, their discriminants are a perfect squares. It follows that there are nonnegative integers a, b satisfying $m^2 + 4n = a^2, m^2 - 4n = b^2$. This gives

$$2m^2 = a^2 + b^2, 8n = a^2 - b^2.$$

Note that if 3 divides m, then 3 divides a and b, and hence, 3 divides n. Also note that if $m \equiv \pm 1 \pmod 3$ holds, then a^2, b^2 are congruent to 0 modulo 3, and hence, 3 divides n.

If m is an odd integer, then a is odd, and hence, m^2, a^2 are congruent to 1 modulo 8, which gives that 8 divides $4n$, implying n is even. If m is even, then so are the integers a and b, and we have

$$2n = \left(\frac{a}{2}\right)^2 - \left(\frac{b}{2}\right)^2,$$

which implies that the integers $a/2, b/2$ are of the same parity, and hence, 4 divides $(a/2)^2 - (b/2)^2$, implying that n is even.

We conclude that n is divisible by 6. ■

References

[**Che25**] Evan Chen. *The OTIS Excerpts*. Available at https://web.evanchen.cc/excerpts.html. 2025, pp. vi+289 (cited p. 1)