Orders

MOPSS

3 June 2025

Mathematics Olympiad

Problem Solving Sessions

MOPSS

Department of Mathematics
IISER Bhopal

https://jpsaha.github.io/MOTP/MOPSS/

Suggested readings

- Evan Chen's advice On reading solutions, available at https://blog.evanchen.cc/2017/03/06/on-reading-solutions/.
- Evan Chen's Advice for writing proofs/Remarks on English, available at https://web.evanchen.cc/handouts/english/english.pdf.
- Notes on proofs by Evan Chen from OTIS Excerpts [Che25, Chapter 1].
- Tips for writing up solutions by Edward Barbeau, available at https://www.math.utoronto.ca/barbeau/writingup.pdf.
- Evan Chen discusses why math olympiads are a valuable experience for high schoolers in the post on Lessons from math olympiads, available at https://blog.evanchen.cc/2018/01/05/lessons-from-math-olympiads/.

List of problems and examples

§1 Orders

Let p be a prime, and a be an integer, not divisible by p. The *order of a modulo p*, denoted by $\mathrm{ord}_p(a)$, is defined to be the smallest positive integer such that $a^{\mathrm{ord}_p(a)} \equiv 1 \bmod p$.

Example 1.1 (Tournament of Towns, India RMO 2014a P3)**.** [**Tao06**, Problem 2.2] [**AE11**, Problem 3.81] Suppose for some positive integers r and s, 2^r is obtained by permuting the digits of 2^s in decimal expansion and $2^r, 2^s$ have same number of digits. Prove that $r = s$.

Solution 1. Since a positive integer is congruent to the sum of its digits modulo 9, it follows that 2^r and 2^s are congruent modulo 9.

Let us consider the case that $r < s$. Note that 9 divides $2^{s-r} - 1$. Since the order of 2 modulo 9 is equal to 6, it follows that 6 divides $s - r$, and hence $2^s \geq 64 \cdot 2^r$, which is impossible. This shows that $r \geq s$ holds. Similarly, it also follows that $s \geq r$ holds. This proves that $s = r$, as required. ■

Example 1.2 (Mathematical Ashes 2011 P2)**.** Find all pairs (m, n) of nonnegative integers for which

$$m^2 + 2 \cdot 3^n = m(2^{n+1} - 1).$$

Walkthrough —

(a) Let m, n be nonnegative integers satisfying the given equation. Considering the roots of $x^2 - x(2^{n+1} - 1) + 2 \cdot 3^n$, it follows that

$$3^k + 2 \cdot 3^\ell = 2^{n+1} - 1$$

holds, for some nonnegative integers k, ℓ satisfying $k + \ell = n$.

(b) Show that if $n \geq 6$, then $\min\{k, \ell\} \geq 2$ holds. Note that

$$3^k < 2^{n+1} < 9^{(n+1)/3}$$

holds, implying $k < 2(n+1)/3$. Also note that

$$2 \cdot 3^\ell < 2^{n+1} < 2 \cdot 3^{2n/3}$$

holds, implying $\ell < 2n/3$. Using $k + \ell = n$, it follows that

$$k > \frac{n-2}{3}, \ell > \frac{n-2}{3}.$$

(c) Let us consider the case[a] that $n \geq 6$. Note that $m := \min\{k, \ell\} \geq 2$ holds.

(i) Note that 9 divides $2^{n+1} - 1$, and show that 6 divides $n+1$. Writing $n + 1 = 6j$ yields

$$2^{n+1} - 1 = (4^j - 1)(4^{2j} + 4^j + 1) = (2^j - 1)(2^j + 1)((4^j - 1)^2 + 3 \cdot 4^j).$$

(ii) Noting that $(4^j - 1)^2 + 3 \cdot 4^j$ is divisible by 3, but not by 9, and that the integers $2^j - 1, 2^j + 1$ are coprime, conclude that 3^{m-1} divides one of $2^j - 1, 2^j + 1$.

(iii) Prove that

$$3^{m-1} \leq 2^j + 1 \leq 3^j = 3^{\frac{n+1}{6}},$$

implying

$$m - 1 \leq \frac{n+1}{6}.$$

(iv) Conclude that

$$\frac{n-2}{3} - 1 < m - 1 \leq \frac{n+1}{6}.$$

holds.

(v) This yields $n < 11$, contradicting $n \geq 6$ and 6 divides $n + 1$.

(d) It remains to consider the case $n \leq 5$.

[a]It also suffices to assume that $n \geq 5$ holds to obtain $m \geq 2$.

References

[**AE11**] Titu Andreescu and Bogdan Enescu. *Mathematical Olympiad treasures*. Second. Birkhäuser/Springer, New York, 2011, pp. viii+253. ISBN: 978-0-8176-8252-1; 978-0-8176-8253-8 (cited p. 2)

[**Che25**] Evan Chen. *The OTIS Excerpts*. Available at https://web.evanchen.cc/excerpts.html. 2025, pp. vi+289 (cited p. 1)

[**Tao06**] Terence Tao. *Solving mathematical problems*. A personal perspective. Oxford University Press, Oxford, 2006, pp. xii+103. ISBN: 978-0-19-920560-8; 0-19-920560-4 (cited p. 2)