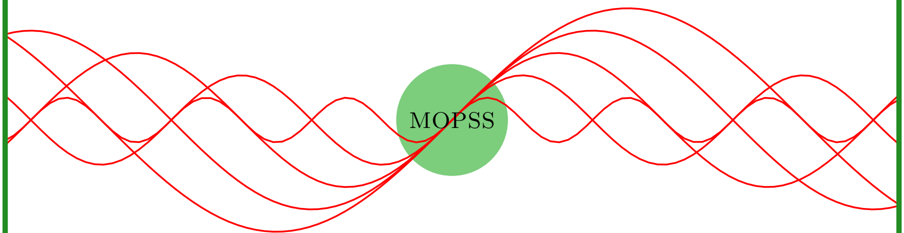
Infinite descent

MOPSS

Mathematics Olympiad

Problem Solving Sessions



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https://jpsaha.github.io/MOTP/MOPSS/

Suggested readings

- Evan Chen's advice On reading solutions, available at https://blog.evanchen.cc/2017/03/06/on-reading-solutions/.
- Evan Chen's Advice for writing proofs/Remarks on English, available at https://web.evanchen.cc/handouts/english/english.pdf.
- Notes on proofs by Evan Chen from OTIS Excerpts [**Che25**, Chapter 1].
- Tips for writing up solutions by Edward Barbeau, available at https://www.math.utoronto.ca/barbeau/writingup.pdf.
- Evan Chen discusses why math olympiads are a valuable experience for high schoolers in the post on Lessons from math olympiads, available at https://blog.evanchen.cc/2018/01/05/lessons-from-math-olympiads/.

List of problems and examples

§1 Infinite descent

Exercise 1.1 (Moscow Mathematical Olympiad First Round 1949 Grades 7–8 P3). [**PK74**, Problem 52.3] Show that the only solution of the equation

$$x^2 + y^2 + z^2 = 2xyz$$

for x, y, z in integers is $x = y = z = 0$.

Exercise 1.2 (Kürschák Competition 1983 P1, AoPS). Rational numbers x, y and z satisfy the equation

$$x^3 + 3y^3 + 9z^3 - 9xyz = 0.$$

Prove that $x = y = z = 0$.

Walkthrough —

(a) Show that if a, b, c are integers satisfying

$$a^3 + 3b^3 + 9c^3 = 9abc,$$

then 3 divides a, and $(b, c, a/3)$ also satisfies the above equation.

(b) If (x, y, z) is a non-trivial integer solution to the given equation with $|x| + |y| + |z|$ minimum, show that x is nonzero, and that $y, z, x/3$ is also a solution to the given equation.

Solution 1. Note that if x, y, z are rational numbers satisfying the given equation, then dx, dy, dz also satisfy the equation for any positive integer d. Hence, it suffices to prove that there are no integer solutions to the given equation other than the trivial solution $x = y = z = 0$.

Claim — If (a, b, c) are integers satisfying

$$x^3 + 3y^3 + 9z^3 = 9xyz,$$

then 3 divides a, and $(b, c, a/3)$ also satisfies the above equation.

Proof of the Claim. Note that 3 divides a^3. Since 3 is a prime, it follows that 3 divides a. Using

$$a^3 + 3b^3 + 9c^3 = 9abc,$$

we obtain

$$b^3 + 3c^3 + 9\left(\frac{a}{3}\right)^3 = 9bc\left(\frac{a}{3}\right).$$

This completes the proof of the claim.

□

Let (x, y, z) be a non-trivial integer solution to the given equation with $|x|+|y|+|z|$ minimum. Note that x is nonzero, otherwise, y, z satisfy $y^3+3z^3 = 0$, which is impossible since y, z are integers. By the above claim, it follows that $y, z, x/3$ is also a solution to the given equation. Using that x is nonzero, we obtain

$$|y| + |z| + \left|\frac{x}{3}\right| < |x| + |y| + |z|,$$

which contradicts the minimality of $|x| + |y| + |z|$. This shows that there are no non-trivial integer solutions to the given equation. ■

> **Remark.** The method used in the above solution is known as *infinite descent*. The idea is to show that if there is a non-trivial solution to the given equation, then there is a **smaller** non-trivial solution. This leads to an infinite sequence of smaller and smaller non-trivial solutions, which is impossible for positive integers.

Exercise 1.3 (BStat-BMath 2012 P5, AoPS)**.** Let m be a natural number with digits consisting entirely of 6's and 0's. Prove that m is not the square of a natural number.

Walkthrough —

(a) Note that if any such number is a perfect square, then its last digit cannot be 6, that is, it is not congruent to 6 modulo 10, because no square is congruent to any of $6, 66$ modulo 100.

(b) It follows that if any such number is perfect square, then it is divisible by 100.

(c) Apply induction (on what?). A crucial step would be frame an inductive statement appropriately.

References

[Che25] Evan Chen. *The OTIS Excerpts*. Available at https://web.evanchen.cc/excerpts.html. 2025, pp. vi+289 (cited p. 1)

[**PK74**] G. Pólya and J. Kilpatrick. *The Stanford Mathematics Problem Book: With Hints and Solutions*. Dover books on mathematics. Teachers College Press, 1974. isbn: 9780486469249. url: https://books.google.de/books?id=Q8Gn51gS6RoC (cited p. 2)