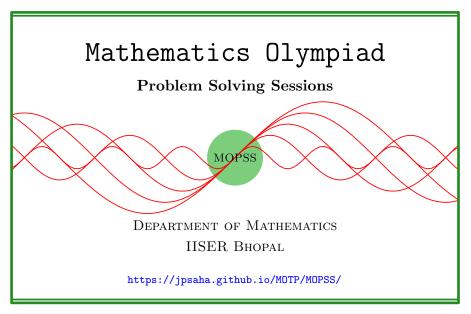
# **Cubic polynomials**

#### MOPSS

26 April 2025



## Suggested readings

- Evan Chen's advice On reading solutions, available at https://blog.evanchen.cc/2017/03/06/on-reading-solutions/.
- Evan Chen's Advice for writing proofs/Remarks on English, available at https://web.evanchen.cc/handouts/english/english.pdf.
- Notes on proofs by Evan Chen from OTIS Excerpts [Che25, Chapter 1].
- Tips for writing up solutions by Edward Barbeau, available at https://www.math.utoronto.ca/barbeau/writingup.pdf.
- Evan Chen discusses why math olympiads are a valuable experience for high schoolers in the post on Lessons from math olympiads, available at https://blog.evanchen.cc/2018/01/05/lessons-from-math-olympiads/.

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# §1 Cubic polynomials

**Example 1.1** (India RMO 2000 P2). Solve the equation  $y^3 = x^3 + 8x^2 - 6x + 8$ , for positive integers x and y.

**Solution 1.** Let x, y be positive integers satisfying the given equation. Since x is a positive integer, it follows that  $8x^2 - 6x + 8$  is positive, and hence,  $y^3 \ge x^3$  holds. This shows that y = x + k for some positive integer k. Substituting y = x + k in the given equation and simplifying, we obtain

$$(3k-8)x^2 + (3k^2+6)x + k^3 - 8 = 0.$$

Since x is positive, it follows that  $k \leq 2$ . If k = 1, then

$$5x^2 - 9x + 7 = 0$$

holds, which shows that  $7 \equiv x(x+1) \pmod{2}$ , which yields  $7 \equiv 0 \pmod{2}$ , which is impossible. This implies that k=2. It follows that

$$2x^2 - 18x = 0$$

which gives x = 9, and consequently, we obtain y = 9 + 2 = 11. Also note that if (k, x) = (2, 9), then

$$(3k-8)x^2 + (3k^2+6)x + k^3 - 8 = 0$$

holds, and it gives

$$(x+k)^3 = x^3 + 8x^2 - 6x + 8,$$

which shows that (x, y) = (9, 11) is a solution to the given equation.

It follows that (9,11) is the only solution of the given equation in the positive integers.

**Remark.** After arriving at the above solution, one can rewrite it to make it brief by observing that

$$x^{3} + 8x^{2} - 6x + 8 - (x+1)^{3} = 5x^{2} - 9x + 7,$$

which is positive for any positive integer x (in fact, it is positive for any real number x), and then concluding that y = x + k for some integer k > 1.

**Example 1.2** (India RMO 2015f P5). Solve the equation  $y^3 + 3y^2 + 3y = x^3 + 5x^2 - 19x + 20$  for positive integers x, y.

**Solution 2.** Let x, y be positive integers satisfying the above equation. Note that the given equation can be rewritten as

$$(y+1)^3 = x^3 + 5x^2 - 19x + 21.$$

Note that

$$5x^2 - 19x + 21 > 5x^2 - 19x > 0$$

holds if  $x \ge 4$ . Also note that  $5x^2 - 19x + 21 > 0$  if x lies in  $\{1, 2, 3\}$ . Since x is a positive integer, it follows that  $5x^2 - 19x + 21 \ge 1$ . This shows that  $(y+1)^3 > x^3$ , and hence, y+1=x+k for some positive integer k. Note that

$$(y+1)^3 = x^3 + 5x^2 - 19x + 21$$

is equivalent to

$$3kx^2 + 3k^2x + k^3 = 5x^2 - 19x + 21,$$

which simplifies to

$$(3k-5)x^2 + (3k^2 + 19)x + (k^3 - 21) = 0.$$

Note that if  $k \geq 2$ , then using x > 0, we obtain

$$(3k-5)x^2 + (3k^2 + 19)x + (k^3 - 21)$$
  
=  $(3k-5)x^2 + 3k^2x + 19(x-1) + k^3 - 2$   
> 1.

This shows that k is equal to 0 or 1. If k = 0, then  $5x^2 - 19x + 21 = 0$  holds, which implies that 21 is even, which is impossible. This shows that k = 1, and hence x = y. It follows that

$$2x^2 - 22x + 20 = 0,$$

and hence x is equal to one of 1, 10. Consequently, we obtain (x, y) is equal to (1, 1) or (10, 10).

Note that for any integers x, y, k satisfying y + 1 = x + k and

$$3kx^2 + 3k^2x + k^3 = 5x^2 - 19x + 21,$$

, we have

$$(y+1)^3 = x^3 + 5x^2 - 19x + 21.$$

It follows that (1,1),(10,10) also satisfy the given equation.

Consequently, the solutions of the given equation over the positive integers are precisely (1,1) and (10,10).

# References

[Che25] EVAN CHEN. The OTIS Excerpts. Available at https://web.evanchen.cc/excerpts.html. 2025, pp. vi+289 (cited p. 1)