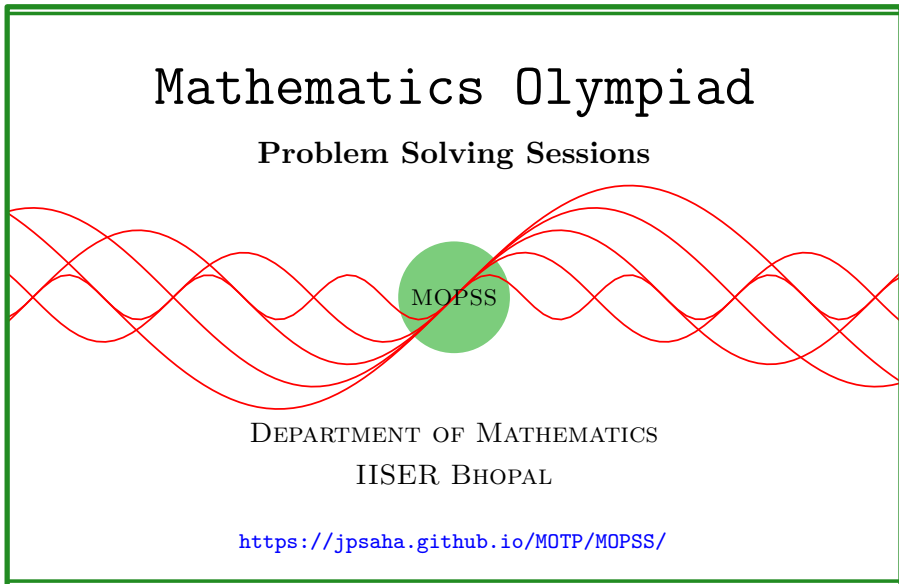


# Binomial coefficients

MOPSS

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## Suggested readings

- Evan Chen's advice [On reading solutions](https://blog.evanchen.cc/2017/03/06/on-reading-solutions/), available at <https://blog.evanchen.cc/2017/03/06/on-reading-solutions/>.
- Evan Chen's [Advice for writing proofs/Remarks on English](https://web.evanchen.cc/handouts/english/english.pdf), available at <https://web.evanchen.cc/handouts/english/english.pdf>.
- [Notes on proofs](#) by Evan Chen from [OTIS Excerpts](#) [[Che25](#), Chapter 1].
- [Tips for writing up solutions](https://www.math.utoronto.ca/barbeau/writingup.pdf) by Edward Barbeau, available at <https://www.math.utoronto.ca/barbeau/writingup.pdf>.
- Evan Chen discusses why [math olympiads are a valuable experience for high schoolers](#) in the post on [Lessons from math olympiads](#), available at <https://blog.evanchen.cc/2018/01/05/lessons-from-math-olympiads/>.

# List of problems and examples

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## §1 Binomial coefficients

**Example 1.1** (AIME 1983 P8, India Pre-RMO 2015 P10(?)). Determine the largest 2-digit prime factor of the integer  $\binom{200}{100}$ .

**Solution 1.** The highest power of any two digit prime  $p$  dividing  $(100!)^2$  is at least  $p^2$ , and hence, if  $p$  is a two digit prime that divides  $\binom{200}{100}$ , then  $p^3$  divides  $200!$ , or equivalently,  $3p \leq 200$ , which implies that  $p \leq 66$ , and this gives  $p \leq 61$ .

Note that 61 is a primes, its highest power dividing  $(100!)^2$  is  $61^2$ . Using  $3 \cdot 61 < 200$ , it follows that  $61^3$  divides  $200!$ . Hence, 61 divides  $\binom{200}{100}$ .

This shows that 61 is the largest 3-digit prime factor of  $\binom{200}{100}$ . ■

**Example 1.2** (India RMO 1992 P3). Determine the largest 3-digit prime factor of the integer  $\binom{2000}{1000}$ .

**Solution 2.** Suppose there is a 3-digit prime  $p$  which divides the integer  $\binom{2000}{1000}$ . Note that

$$\binom{2000}{1000} = \frac{2000 \cdot 1999 \cdot 1998 \cdot \dots \cdot 1001}{1000 \cdot 999 \cdot 998 \cdot \dots \cdot 1}$$

holds. If  $p > 500$ , then using the fact that  $p$  divides  $\binom{2000}{1000}$ , it follows that  $3p \leq 2000$ , which implies  $p \leq 666$ , and hence,  $p \leq 661$ . This shows that any 3-digit prime divisor of  $\binom{2000}{1000}$ , larger than 500, is at most 661.

Note that 661 is a 3-digit prime, and it divides  $1000!$ . Since  $1000 > 661 > 500$  holds, it follows that  $2000 > 2 \cdot 661 \geq 1001$ . Also note that  $3 \cdot 661 < 2000$ . This implies that the highest power of 661 dividing  $1000!$  is  $661$ , and the highest power of 661 dividing  $2000 \cdot 1999 \cdot 1998 \cdot \dots \cdot 1001$  is at least  $661^2$ . Consequently, 661 divides  $\binom{2000}{1000}$ .

This proves that 661 is the largest 3-digit prime factor of the integer  $\binom{2000}{1000}$ . ■

## References

- [Che25] EVAN CHEN. *The OTIS Excerpts*. Available at <https://web.evanchen.cc/excerpts.html>. 2025, pp. vi+289 (cited p. 1)