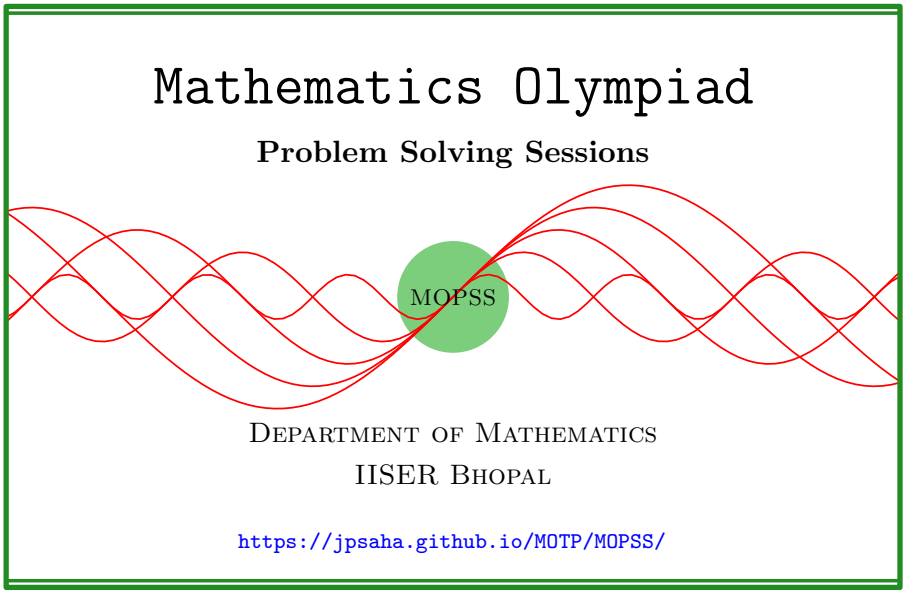
MOPSS

17 October 2026



Suggested readings

- Evan Chen's advice On reading solutions, available at https://blog.evanchen.cc/2017/03/06/on-reading-solutions/.
- Evan Chen's Advice for writing proofs/Remarks on English, available at https://web.evanchen.cc/handouts/english/english.pdf.
- Notes on proofs by Evan Chen from OTIS Excerpts [**Che25**, Chapter 1].
- Tips for writing up solutions by Edward Barbeau, available at https://www.math.utoronto.ca/barbeau/writingup.pdf.
- Evan Chen discusses why math olympiads are a valuable experience for high schoolers in the post on Lessons from math olympiads, available at https://blog.evanchen.cc/2018/01/05/lessons-from-math-olympiads/.

Exercise 1 (RMO 2023b P3,). For any natural number n, expressed in base 10, let $s(n)$ denote the sum of all its digits. Find all natural numbers m and n such that $m < n$ and

$$(s(n))^2 = m \text{ and } (s(m))^2 = n.$$

Solution 1. Let m, n be natural numbers satisfying the given conditions. Note that

$$n^2 \equiv m \bmod 9, m^2 \equiv n \bmod 9,$$

which implies that m is congruent to one of $0, 1$ modulo 3, and so is n, and consequently, it follows that

$$m \equiv n \bmod 3.$$

Suppose m has d digits. Note that $m \geq 10^{d-1}$ and $s(m) \leq 9d$ holds. Consequently, $10^{d-1} > (9d)^2$ would yield

$$n > m \geq 10^{d-1} > (9d)^2 \geq s(m)^2 = n,$$

which is impossible. Observe that $10^{5-1} > (9 \cdot 5)^2$ holds, and if k is a positive integer satisfying $10^{k-1} > (9k)^2$, then

$$\begin{aligned} \frac{10^k}{(9(k+1))^2} &> \frac{10(9k)^2}{(9(k+1))^2} \\ &= \frac{10}{(1+\frac{1}{k})^2} \\ &> 1 \end{aligned}$$

holds. By induction, it follows that $d \leq 4$, that is, m has at most 4 digits.

This gives

$$s(m) \leq 9d \leq 36,$$

and hence, it follows that

$$n = s(m)^2 \leq 36^2 = 1296.$$

This yields

$$s(n) \leq \max\{s(1296), s(1199), s(1099), s(999)\} = \max\{18, 20, 19, 27\} = 27,$$

which gives

$$m = s(n)^2 \leq 27^2 = 729.$$

We get

$$s(m) \leq \max\{s(729), s(719), s(709), s(699)\} = \max\{18, 17, 16, 24\} = 24,$$

which shows that

$$n = s(m)^2 \leq 24^2 = 576.$$

This gives

$$s(n) \leq \max\{s(576), s(569), s(559), \ldots, s(519), s(509), s(499)\} = 22,$$

which yields

$$m = s(n)^2 \leq 22^2 = 484.$$

This shows that

$$s(m) \leq \max\{s(484), s(479), s(399)\} = 21,$$

which gives

$$n = s(m)^2 \leq 21^2 = 441.$$

It follows that

$$s(n) \leq \max\{s(441), s(400), s(361), s(324), s(289)\} = \max\{9, 4, 10, 9, 19\} = 19,$$

and hence, we get

$$m = s(n)^2 \leq 19^2 = 361.$$

This gives

$$s(m) \leq \max\{s(361), s(324), s(289)\} = 19,$$

which yields

$$n \leq 19^2,$$

and using the inequality $m < n$, the congruence $m \equiv n \bmod 3$, and that m is a perfect square, we obtain

$$m \leq 17^2.$$

Note that

$$s(17^2)^2 = s(289)^2 = 19^2 = 361, \quad s(361)^2 = 100 \neq 17^2$$

holds, which implies that

$$m \leq 16^2,$$

and hence, we obtain

$$s(m) \leq \max\{s(256), s(225), s(196), s(169), s(81), s(64), s(49)\} = 16.$$

Consequently, we obtain

$$n \leq 16^2,$$

and using the inequality $m < n$, the congruence $m \equiv n \bmod 3$, and that m is a perfect square, we get

$$m \leq 14^2.$$

Observe that

$$s(14^2)^2 = s(196)^2 = 16^2 = 256, \quad s(256)^2 = 13^2 \neq 14^2$$

holds, which yields

$$m \leq 13^2.$$

Note that $(m, n) = (13^2, 16^2)$ is a solution to the given equation. If $8^2 \leq m \leq 12^2$, then

$$s(m) \leq \max\{s(64), s(81), s(100), s(121), s(144)\} = 9,$$

which gives $n \leq 81$, and hence (m, n) is equal to $(8^2, 9^2)$, which is impossible. It follows that $m \leq 7^2$. Observe that $s(m)^2 = n, s(n)^2 = m$ fails to hold if m is any of $1^2, 2^2, 3^2, 4^2, 5^2, 6^2, 7^2$.

We conclude that $(m, n) = (13^2, 16^2)$ is the only solution to the given equations. ■

Remark. The above solution to Exercise 1 makes an effort to defer computing the (decimal expansions of the) perfect squares. It turns out that without deferring it, we may arrive at the solution in fewer steps.

After obtaining $n \leq 36^2 = 1296$, one may compute the sum of the digits of the perfect squares lying between 11^2 and 36^2, as recorded in the table below.

k	k^2	$s(k^2)$
36	1296	18
35	1225	10
34	1156	13
33	1089	18
32	1024	7
31	961	16
30	900	9
29	841	13
28	784	19
27	729	18
26	676	19
25	625	13
24	576	18
23	529	16
22	484	16
21	441	9
20	400	4
19	361	10
18	324	9
17	289	19
16	256	13
15	225	9

14	196	16
13	169	16
12	144	9
11	121	4

This yields

$$s(n) \leq 19,$$

which gives

$$m = s(n)^2 \leq 19^2 = 361.$$

We get

$$s(m) \leq 19,$$

which shows that

$$n = s(m)^2 \leq 19^2 = 361.$$

Since $m \equiv n \bmod 3$, we obtain

$$m \leq 17^2 = 289.$$

Note that

$$s(17^2)^2 = s(289)^2 = 19^2 = 361, \quad s(361)^2 = 100 \neq 17^2$$

holds, which implies that

$$m \leq 16^2,$$

and hence, we obtain

$$s(m) \leq \max\{s(256), s(225), s(196), s(169), s(81), s(64), s(49)\} = 16.$$

Consequently, we obtain

$$n \leq 16^2,$$

and this gives

$$m \leq 15^2.$$

Observe that

$$s(15^2)^2 = s(225)^2 = 81, \quad s(81)^2 = 81 \neq 15^2,$$

$$s(14^2)^2 = s(196)^2 = 16^2 = 256, \quad s(256)^2 = 13^2 \neq 14^2$$

holds, which yields

$$m \leq 13^2.$$

Note that $(m, n) = (13^2, 16^2)$ is a solution to the given equation. If $8^2 \leq m \leq 12^2$, then

$$s(m) \leq \max\{s(64), s(81), s(100), s(121), s(144)\} = 9,$$

which gives $n \leq 81$, and hence (m, n) is equal to $(8^2, 9^2)$, which is impossible. It follows that $m \leq 7^2$. Observe that $s(m)^2 = n, s(n)^2 = m$ fails to hold if m is any of $1^2, 2^2, 3^2, 4^2, 5^2, 6^2, 7^2$.

We conclude that $(m, n) = (13^2, 16^2)$ is the only solution to the given equations.

Exercise 2 (RMO 2015c P3,). Let $P(x)$ be a nonconstant polynomial whose coefficients are positive integers. If $P(n)$ divides $P(P(n) - 2015)$ for all natural numbers n, then prove that $P(-2015) = 0$.

Summary — In absolute value, a higher degree polynomial dominates a smaller degree polynomial at arguments which are large enough in absolute value.

Solution 2. Let $Q(x), R(x)$ be polynomials with rational coefficients such that

$$P(P(x) - 2015) = P(x)Q(x) + R(x)$$

and $R(x) = 0$ or $\deg R(x) < \deg P(x)$. Note that $P(n)$ is positive for all integer $n \geq 1$ since the coefficients of $P(x)$ are positive integers. By the given condition, it follows that $P(n)$ divides $R(n)$ for any integer $n \geq 1$.

Claim — Let $f(x), g(x)$ be two polynomials with real coefficients. Suppose $f(x)$ is a nonconstant polynomial with a positive leading coefficient, and $\deg g(x) < \deg f(x)$. Then there exists an integer $n_0 \geq 1$ such that

$$f(n) > g(n)$$

for any $n \geq n_0$.

Proof of the Claim. Note that it suffices to proves the Claim if $f(x)$ is a monomial, that is, a power of x. Indeed, write $f(x) = a_d x^d + a_{d-1}x^{d-1} + \cdots + a_0$ with $a_0, \ldots, a_d \in \mathbb{R}$ and d denoting the degree of f. Also write $g(x) = b_e x^e + b_{e-1}x^{e-1} + \cdots + b_0$ with $b_0, \ldots, b_e \in \mathbb{R}$ and e denoting the degree of g. Noting that $a_d > 0$, it follows that for a positive integer n, the inequality

$$a_d n^d + a_{d-1}n^{d-1} + \cdots + a_0 > b_e n^e + b_{e-1}n^{e-1} + \cdots + b_0$$

holds if

$$a_d n^d > \frac{a_{d-1}}{a_d}n^{d-1} + \cdots + \frac{a_0}{a_d} + \frac{b_e}{a_d}n^e + \frac{b_{e-1}}{a_d}n^{e-1} + \cdots + \frac{b_0}{a_d}$$

is satisfied, which can be concluded provided the Claim is known in the case when f is a monomial.

Let us assume that f is a monomial. Write $f(x) = x^d$ where d denotes the degree of f, and write $g(x) = b_e x^e + b_{e-1}x^{e-1} + \cdots + b_0$ with $b_0, \ldots, b_e \in \mathbb{R}$ and e denoting the degree of g. For any integer n, note that

$$\begin{aligned} f(n) - g(n) = &\left(\frac{1}{e+1}n^d - b_e n^e\right) \\ &+ \left(\frac{1}{e+1}n^d - b_{e-1}n^{e-1}\right) \\ &+ \cdots + \left(\frac{1}{e+1}n^d - b_0\right) \end{aligned}$$

$$\geq \left(\frac{1}{e+1}n^d - |b_e|n^e\right)$$
$$+ \left(\frac{1}{e+1}n^d - |b_{e-1}|n^{e-1}\right)$$
$$+ \cdots + \left(\frac{1}{e+1}n^d - |b_0|\right).$$

Since $d \geq e$, it follows that there exists an integer $n_0 \geq 1$ such that

$$\frac{1}{e+1}n^d - |b_e|n^e \frac{1}{e+1}n^d - |b_{e-1}|n^{e-1}, \ldots, \frac{1}{e+1}n^d - |b_0|$$

are positive for any $n \geq n_0$. This proves the Claim. □

By the above Claim, it follows that $R(x)$ is the zero polynomial. This implies that

$$P(P(x) - 2015) = P(x)Q(x).$$

Since $P(x)$ is a nonconstant polynomial, it has a root z in $\mathbb{C}$. Substituting $x = z$ yields

$$P(-2015) = 0.$$

■

Here is an alternate solution.

Solution 3. Since $P(x)$ is a polynomial with integer coefficients, it follows that for any integers a, b, the difference $P(a) - P(b)$ is divisible by $a - b$. This shows that

$$P(P(n) - 2015) - P(-2015)$$

is divisible by $P(n)$ for all natural numbers n. Since $P(n)$ divides $P(P(n)-2015)$ for all natural numbers n, it follows that $P(n)$ divides $P(-2015)$ for all natural numbers n. Let d denote the degree of $P(x)$, and a_d denote the leading coefficient of $P(x)$. Since $P(x)$ has positive integer coefficients, it follows that

$$P(n) \geq a_d n^d$$

holds for any natural number n. Taking $k = |P(-2015)|$, we have that $P(k+1)$ divides k. Since $P(k+1) \geq a_d(k+1)^d > k$ and k is a nonnegative integer, we conclude that $k = 0$, which implies that $P(-2015) = 0$. ■

Exercise 3 (RMO 2019a P3,). Let a, b, c be positive real numbers such that $a + b + c = 1$. Prove that

$$\frac{a}{a^2+b^3+c^3} + \frac{b}{b^2+a^3+c^3} + \frac{c}{c^2+a^3+b^3} \leq \frac{1}{5abc}.$$

Walkthrough — **Homogenize** the denominator and apply the AM-GM inequality.

Solution 4. Note that

$$\begin{aligned}\frac{a}{a^2+b^3+c^3} &= \frac{a}{a^2(a+b+c)+b^3+c^3}\\ &\le \frac{a}{5\sqrt[5]{a^6\cdot abc\cdot b^3\cdot c^3}}\\ &= \frac{1}{5\sqrt[5]{a^2b^4c^4}}\\ &= \frac{1}{5abc}\left(a^3bc\right)^{1/5}\\ &\le \frac{3a+b+c}{25abc}\end{aligned}$$

holds. Similarly, it follows that

$$\begin{aligned}\frac{b}{b^2+a^3+c^3} &\le \frac{3b+a+c}{25abc},\\ \frac{c}{c^2+a^3+b^3} &\le \frac{3c+a+b}{25abc}.\end{aligned}$$

Adding the above inequalities, we obtain

$$\begin{aligned}&\frac{a}{a^2+b^3+c^3}+\frac{b}{b^2+a^3+c^3}+\frac{c}{c^2+a^3+b^3}\\ &\le \frac{3a+b+c}{25abc}+\frac{3b+a+c}{25abc}+\frac{3c+a+b}{25abc}\\ &= \frac{a+b+c}{5abc}\\ &= \frac{1}{5abc}.\end{aligned}$$

■

References

[**Che25**] Evan Chen. *The OTIS Excerpts*. Available at https://web.evanchen.cc/excerpts.html. 2025, pp. vi+289 (cited p. 1)