MOPSS

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Mathematics Olympiad

Problem Solving Sessions

Department of Mathematics

IISER Bhopal

https://jpsaha.github.io/MOTP/MOPSS/

Suggested readings

- Evan Chen's advice On reading solutions, available at https://blog.evanchen.cc/2017/03/06/on-reading-solutions/.
- Evan Chen's Advice for writing proofs/Remarks on English, available at https://web.evanchen.cc/handouts/english/english.pdf.
- Notes on proofs by Evan Chen from OTIS Excerpts [**Che25**, Chapter 1].
- Tips for writing up solutions by Edward Barbeau, available at https://www.math.utoronto.ca/barbeau/writingup.pdf.
- Evan Chen discusses why math olympiads are a valuable experience for high schoolers in the post on Lessons from math olympiads, available at https://blog.evanchen.cc/2018/01/05/lessons-from-math-olympiads/.

List of problems and examples

§1

Exercise 1.1 (Serbia IMO TST 2024 Day 1 P1, AoPS)**.** Three coins are placed at the origin of a Cartesian coordinate system. On one move one removes a coin placed at some position (x, y) and places three new coins at $(x+1, y)$, $(x, y+1)$ and $(x+1, y+1)$. Prove that after finitely many moves, there will exist two coins placed at the same point.

> **Remark.** The problem seems to ask that after any number of finitely many moves, there will exist two coins placed at the same point.

> **Walkthrough** —
> **(a)**

Solution 1. Let S denote the sum of $n_{(x,y)} r^{x+y}$ over all points (x, y), where there is a coin at (x, y) at some stage of the process, and $n_{(x,y)}$ denotes the number of coins at (x, y) at that stage, and r is a real number to be determined later. If we remove a coin at (x, y) and place three coins at $(x+1, y)$, $(x, y+1)$ and $(x+1, y+1)$, then S changes to

$$S - r^{x+y} + r^{x+1+y} + r^{x+y+1} + r^{x+1+y+1} = S + r^{x+y}(2r + r^2 - 1).$$

If we choose r such that $2r + r^2 - 1 = 0$, then S does not change after any move. Let us take $r = \sqrt{2} - 1$. Since it is root of the equation $2r + r^2 - 1 = 0$, it follows that S is invariant under the moves. Initially, $S = 3$ since there are three coins at the origin.

If there are no two coins at the same point at some stage of the process after finitely many moves, then S is the sum of r^{x+y} over all points (x, y) where there is a coin, and this implies that

$$\begin{aligned} 3 &= S \\ &< \sum_{(x,y)\in\mathbb{Z}^2_{\geq 0}} r^{x+y} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{x=0}^{\infty}\sum_{y=0}^{\infty} r^{x+y} \\
&= \left(\sum_{x=0}^{\infty} r^x\right)^2 \\
&= \frac{1}{(1-r)^2} \\
&= \left(1+\frac{1}{\sqrt{2}}\right)^2 \\
&= \frac{3}{2}+\sqrt{2} \\
&< 3,
\end{aligned}
$$

which is a contradiction. This proves that after any number of finitely many moves, there will exist two coins placed at the same point. ■

Exercise 1.2 (Chip-firing game)**.** Assume that on the Cartesian plane, four chips are placed at the origin. In each step, you may choose a lattice point (x, y) having at least a chip on it, remove one chip from (x, y), and place one chip each at $(x + 1, y)$ and one at $(x, y + 1)$. Show that after a finite number of steps, there are at least two chips at some lattice point.

> **Remark.** The problem seems to ask that after any number of finitely many moves, there will exist two coins placed at the same point.

> **Walkthrough** —
>
> **(a)**

Solution 2. Let S denote the sum of $n_{(x,y)} r^{x+y}$ over all points (x, y), where there is a chip at (x, y) at some stage of the process, and $n_{(x,y)}$ denotes the number of chips at (x, y) at that stage, and r is a real number to be determined later. If we remove a chip at (x, y) and place one chip each at $(x + 1, y)$ and $(x, y + 1)$, then S changes to

$$S - r^{x+y} + r^{x+1+y} + r^{x+y+1} = S + r^{x+y}(2r - 1).$$

If we choose $r = \frac{1}{2}$, then S does not change after any move. Initially, $S = 4$ since there are four chips at the origin.

If there are no two chips at the same point at some stage of the process after finitely many moves, then S is the sum of r^{x+y} over all points (x, y) where there is a chip, and this implies that

$$4 = S$$

$$\begin{aligned}
&< \sum_{(x,y)\in\mathbb{Z}^2_{\geq 0}} r^{x+y} \\
&= \sum_{x=0}^{\infty}\sum_{y=0}^{\infty} r^{x+y} \\
&= \left(\sum_{x=0}^{\infty} r^x\right)^2 \\
&= \frac{1}{(1-r)^2} \\
&= 4,
\end{aligned}$$

which is a contradiction. Thus, there are at least two chips at some lattice point after a finite number of steps. ■

Exercise 1.3 (Swiss Mathematical Olympiad 2023 Final Round Day 1 P2, AoPS)**.** The wizard Albus and Brian are playing a game on a square of side length $2n+1$ meters surrounded by lava. In the centre of the square there sits a toad. In a turn, a wizard chooses a direction parallel to a side of the square and enchants the toad. This will cause the toad to jump d meters in the chosen direction, where d is initially equal to 1 and increases by 1 after each jump. The wizard who sends the toad into the lava loses. Albus begins and they take turns. Depending on n, determine which wizard has a winning strategy.

Walkthrough —

(a)

Solution 3. Note that if $n=0$, then Brian wins. Henceforth, we assume that $n \geq 1$. We claim that Brian wins if he follows the strategy of sending the toad in the opposite direction of Albus's previous jump.

Assume for the sake of contradiction that Brian's strategy fails after the kth turn of Albus. Since $n \geq 1$, we have $k \geq 2$. Note that a move of Albus and the subsequent move of Brian together cause a net displacement of the toad by 1 meter. Hence, two steps prior to Brian's kth turn, that is, after Brian's $(k-1)$th turn, the toad is at the edge of the square, otherwise, after next two moves, the toad would still be in the square. Moreover, the toad reaches the edge of the square after at least $2n$ moves, since the toad starts at the centre of the square. Therefore, $k-1 \geq n$, and hence, $k \geq n+1$. On the other hand, during k-th move of Albus, the toad jumps $2k$ meters from the edge of the square, which is more than the side length of the square, and hence, Albus loses. This is a contradiction, and hence, Brian's strategy works. ■

Exercise 1.4 (British Mathematical Olympiad Round 2 2023 P2, AoPS)**.** For an integer $n > 1$, the numbers $1, 2, 3, \ldots, n$ are written in order on a blackboard. The following *moves* are possible:

(i) Take three adjacent numbers x, y, z whose sum is a multiple of 3 and replace them with y, z, x.

(ii) Take two adjacent numbers x, y whose difference is a multiple of 3 and replace them with y, x.

For example we could take: $1, 2, 3, 4 \xrightarrow{\text{(i)}} 2, 3, 1, 4 \xrightarrow{\text{(ii)}} 2, 3, 4, 1$. Find all n such that the initial list can be transformed into $n, 1, 2, \ldots, n-1$ after a finite number of moves.

Walkthrough —

(a)

Solution 4. Let $x_1, x_2, \ldots, x_n$ denote the numbers on the blackboard, in the order they are written, prior to a move. We will show that the modulo 3 congruence class of the following quantity is invariant under the moves:

$$S = \sum_{i=1}^{n} i x_i.$$

For move (i), we have $x_i = x$, $x_{i+1} = y$ and $x_{i+2} = z$ for some $1 \leq i \leq n-2$. Then the change in S is

$$\begin{aligned}
&(iy + (i+1)z + (i+2)x) - (ix + (i+1)y + (i+2)z) \\
&= -y - z + 2x \\
&\equiv -(x+y+z) \pmod 3 \\
&\equiv 0 \pmod 3.
\end{aligned}$$

For move (ii), we have $x_i = x$ and $x_{i+1} = y$ for some $1 \leq i \leq n-1$. Then the change in S is

$$iy + (i+1)x - (ix + (i+1)y) = x - y \equiv 0 \pmod 3.$$

Thus S is invariant modulo 3. Initially, we have

$$\begin{aligned}
S &= 1 \cdot 1 + 2 \cdot 2 + \cdots + n \cdot n \\
&= 1^2 + 2^2 + \cdots + n^2.
\end{aligned}$$

Let $n > 1$ be an integer such that the initial list can be transformed into $n, 1, 2, \ldots, n-1$ after a finite number of moves. Then we also have

$$S = 1 \cdot n + 2 \cdot 1 + 3 \cdot 2 + \cdots + n \cdot (n-1)$$

$$= n + (2^2 - 2) + (3^2 - 3) + \cdots + (n^2 - n)$$
$$\equiv 1^2 + 2^2 + \cdots + n^2 - (1 + 2 + \cdots + n) + n.$$

Since S is invariant modulo 3, we have

$$1^2 + 2^2 + \cdots + n^2 \equiv 1^2 + 2^2 + \cdots + n^2 - (1 + 2 + \cdots + n) + n \pmod 3,$$

which simplifies to

$$1 + 2 + \cdots + n \equiv n \pmod 3.$$

This gives

$$\frac{n(n-1)}{2} \equiv 0 \pmod 3,$$

and hence, either $n \equiv 0 \pmod 3$ or $n \equiv 1 \pmod 3$.

> **Claim** — Let $n > 1$ be an interger satisfying $n \equiv 0 \pmod 3$ or $n \equiv 1 \pmod 3$. Then the initial list can be transformed into $n, 1, 2, \ldots, n-1$ after a finite number of moves.

Proof of the claim. Note that the move (i) can be applied to $1, 2, 3$ to get $2, 3, 1$, and the move (i) can be applied again to $2, 3, 1$ to get $3, 1, 2$. Thus we can transform $1, 2, 3$ into $3, 1, 2$. Similarly, the move (i) can be applied to $1, 2, 3, 4$ to get $1, 3, 4, 2$, and the move (i) can be applied again to $1, 3, 4, 2$ to obtain $1, 4, 2, 3$. Applying the move (ii) to $1, 4, 2, 3$ gives $4, 1, 2, 3$. Thus we can transform $1, 2, 3, 4$ into $4, 1, 2, 3$. Let k be a positive integer such that the list $1, 2, \ldots, n$ can be transformed into $n, 1, 2, \ldots, n-1$ for all $1 < n \leq 3k + 1$ such that 3 divides $n(n-1)$.

Consider the list $1, 2, \ldots, 3k + 3$. Since $4, 5, \ldots, 3k + 3$ are congruent to $1, 2, \ldots, 3k$ modulo 3 respectively, by the induction hypothesis, we can transform $4, 5, \ldots, 3k+3$ into $3k+3, 4, 5, \ldots, 3k+2$. Applying the move (ii) to $1, 2, 3, 3k+3, 4, 5, \ldots, 3k+2$ gives $1, 2, 3k+3, 3, 4, 5, \ldots, 3k+2$. Applying the move (i) to $1, 2, 3k+3$ twice gives $3k+3, 1, 2$. Thus we can transform $1, 2, \ldots, 3k+3$ into $3k+3, 1, 2, 3, 4, \ldots, 3k+2$.

Consider the list $1, 2, \ldots, 3k + 4$. Since $4, 5, \ldots, 3k + 4$ are congruent to $1, 2, \ldots, 3k+1$ modulo 3 respectively, by the induction hypothesis, we can transform $4, 5, \ldots, 3k+4$ into $3k+4, 4, 5, \ldots, 3k+3$. Applying move (i) to $1, 2, 3, 3k+4$ gives $1, 2, 3k+4, 3$. Applying move (i) to $1, 2, 3k+4$ twice gives $3k+4, 1, 2$. Thus we can transform $1, 2, \ldots, 3k+4$ into $3k+4, 1, 2, 3, 4, \ldots, 3k+3$.

This completes the induction step, and hence the claim holds by induction. □

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Exercise 1.5 (Canadian Mathematical Olympiad 2024 P2, AoPS)**.** Jane writes down 2024 natural numbers around the perimeter of a circle. She wants the 2024 products of adjacent pairs of numbers to be exactly the set $\{1!, 2!, \ldots, 2024!\}$. Can she accomplish this?

Walkthrough —

(a) Show that to have such a configuration, the product $1! \cdot 2! \cdot \ldots \cdot 2024!$ must be a perfect square.

(b) Show that $1! \cdot 2! \cdot \ldots \cdot 2024!$ is not a perfect square.

Solution 5. We claim that Jane cannot accomplish this. Assume for the sake of contradiction that she can. Let the numbers be $a_1, a_2, \ldots, a_{2024}$ in clockwise order, and let $a_{2025} = a_1$. Then for some permutation $k_1, k_2, \ldots, k_{2024}$ of $1, 2, \ldots, 2024$, we have $a_i a_{i+1} = k_i!$ for any $1 \leq i \leq 2024$. It follows that

$$(a_1 a_2 a_3 \cdots a_{2024})^2 = 1! \cdot 2! \cdot \ldots \cdot 2024!.$$

Note that 1009 divides each of $1009!, 1010!, \ldots, 2024!$ but does not divide any of $1!, 2!, \ldots, 1008!$. Moreover, 1009^2 does not divide any of $1009!, 1010!, \ldots, 2017!$. Also note that 1009^2 divides $2018!, 2019!, \ldots, 2024!$, and 1009^3 does not divide any of $2018!, 2019!, \ldots, 2024!$. This implies that the largest nonnegative integer r such that 1009^r divides $1! \cdot 2! \cdot \ldots \cdot 2024!$ is equal to

$$(2024 - 1008) + (2024 - 2017),$$

which is odd. This implies that $1! \cdot 2! \cdot \ldots \cdot 2024!$ is not a perfect square, contradicting the earlier equation. Hence, Jane cannot accomplish this. ■

Remark. Note that no prime larger than 1009 helps to show that $1! \cdot 2! \cdot \ldots \cdot 2024!$ is not a perfect square.

References

[**Che25**] Evan Chen. *The OTIS Excerpts*. Available at https://web.evanchen.cc/excerpts.html. 2025, pp. vi+289 (cited p. 1)