MOPSS

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Mathematics Olympiad

Problem Solving Sessions

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https://jpsaha.github.io/MOTP/MOPSS/

Suggested readings

- Evan Chen's advice On reading solutions, available at https://blog.evanchen.cc/2017/03/06/on-reading-solutions/.
- Evan Chen's Advice for writing proofs/Remarks on English, available at https://web.evanchen.cc/handouts/english/english.pdf.
- Notes on proofs by Evan Chen from OTIS Excerpts [**Che25**, Chapter 1].
- Tips for writing up solutions by Edward Barbeau, available at https://www.math.utoronto.ca/barbeau/writingup.pdf.
- Evan Chen discusses why math olympiads are a valuable experience for high schoolers in the post on Lessons from math olympiads, available at https://blog.evanchen.cc/2018/01/05/lessons-from-math-olympiads/.

List of problems and examples

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Exercise 1.1 (All-Russian Mathematical Olympiad 2007 Grade 10 Day 1 P2, AoPS, by A. Khrabrov)**.** Consider a polynomial $P(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$. Let m denote the minimum of the integers

$$a_0, a_0 + a_1, \ldots, a_0 + a_1 + \cdots + a_n.$$

Prove that $P(x) \geq mx^n$ for all $x \geq 1$.

Walkthrough —

(a)

Solution 1. Note that

$$\begin{aligned} P(x) &= a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n \\ &= a_0(x^n - x^{n-1}) + (a_0 + a_1)(x^{n-1} - x^{n-2}) \\ &\quad + \cdots + (a_0 + a_1 + \cdots + a_{n-1})(x - 1) + (a_0 + a_1 + \cdots + a_n) \end{aligned}$$

holds, which implies that

$$\begin{aligned} &P(x) - mx^n \\ &= a_0(x^n - x^{n-1}) + (a_0 + a_1)(x^{n-1} - x^{n-2}) \\ &\quad + \cdots + (a_0 + a_1 + \cdots + a_{n-1})(x - 1) + (a_0 + a_1 + \cdots + a_n) \\ &\quad - m(x^n - x^{n-1}) - m(x^{n-1} - x^{n-2}) - \cdots - m(x - 1) - m \\ &= (a_0 - m)(x^n - x^{n-1}) + (a_0 + a_1 - m)(x^{n-1} - x^{n-2}) \\ &\quad + \cdots + (a_0 + a_1 + \cdots + a_{n-1} - m)(x - 1) + (a_0 + a_1 + \cdots + a_n - m). \end{aligned}$$

It follows that for any $x \geq 1$,

$$P(x) \geq mx^n.$$

■

Exercise 1.2 (Canadian Mathematical Olympiad 2021 P2, AoPS)**.** Let $n \geq 2$ be some fixed positive integer and suppose that $a_1, a_2, \ldots, a_n$ are positive real numbers satisfying

$$a_1 + a_2 + \cdots + a_n = 2^n - 1.$$

Find the minimum possible value of

$$\frac{a_1}{1} + \frac{a_2}{1+a_1} + \frac{a_3}{1+a_1+a_2} + \cdots + \frac{a_n}{1+a_1+a_2+\cdots+a_{n-1}}.$$

Walkthrough —

(a)

Solution 2. Note that

$$\begin{aligned}
&\frac{a_1}{1} + \frac{a_2}{1+a_1} + \frac{a_3}{1+a_1+a_2} + \cdots + \frac{a_n}{1+a_1+a_2+\cdots+a_{n-1}} \\
&= \frac{1+a_1}{1} + \frac{1+a_1+a_2}{1+a_1} + \frac{1+a_1+a_2+a_3}{1+a_1+a_2} \\
&\quad + \cdots + \frac{1+a_1+a_2+\cdots+a_n}{1+a_1+a_2+\cdots+a_{n-1}} - n \\
&\geq n\sqrt[n]{1+a_1+a_2+\cdots+a_n} - n \\
&= n\sqrt[n]{2^n} - n \\
&= n.
\end{aligned}$$

This shows that the minimum possible value of the given expression is at least n.

Also note that if $a_i = 2^{i-1}$ for all $1 \leq i \leq n$, then $a_1 + a_2 + \cdots + a_n = 2^n - 1$ holds, and

$$\begin{aligned}
&\frac{a_1}{1} + \frac{a_2}{1+a_1} + \frac{a_3}{1+a_1+a_2} + \cdots + \frac{a_n}{1+a_1+a_2+\cdots+a_{n-1}} \\
&= \frac{1+a_1}{1} + \frac{1+a_1+a_2}{1+a_1} + \frac{1+a_1+a_2+a_3}{1+a_1+a_2} \\
&\quad + \cdots + \frac{1+a_1+a_2+\cdots+a_n}{1+a_1+a_2+\cdots+a_{n-1}} - n \\
&= 2n - n \\
&= n.
\end{aligned}$$

This proves that the minimum possible value of the given expression is equal to n. ■

Exercise 1.3 (Canadian Mathematical Olympiad 2019 P2, AoPS)**.** Let a and b be positive integers such that $a+b^3$ is divisible by $a^2+3ab+3b^2-1$. Prove that $a^2+3ab+3b^2-1$ is divisible by the cube of an integer greater than 1.

> **Walkthrough** —
>
> **(a)**

Solution 3. Note that

$$\begin{aligned}(a+b)^3 &= a(a^2+3ab+3b^2)+b^3\\ &= a(a^2+3ab+3b^2-1)+a+b^3\end{aligned}$$

holds. Since $a+b^3$ is divisible by $a^2+3ab+3b^2-1$, it follows that $(a+b)^3$ is also divisible by $a^2+3ab+3b^2-1$. Assume for the sake of contradiction that $a^2+3ab+3b^2-1$ is not divisible by the cube of an integer greater than 1. It follows that $a^2+3ab+3b^2-1$ divides $(a+b)^2$. This implies that

$$a^2+2ab+b^2 \geq a^2+3ab+3b^2-1,$$

which simplifies to

$$ab+2b^2 \leq 1,$$

which is impossible since a and b are positive integers. This proves that $a^2+3ab+3b^2-1$ is divisible by the cube of an integer greater than 1. ■

Exercise 1.4 (All-Russian Mathematical Olympiad 2007 Grade 11 Day 2 P6, AoPS, by N. Agakhanov, I. Bogdanov)**.** Do there exist nonzero reals a, b, c such that, for any $n > 3$, there exists a polynomial $P_n(x) = x^n + \cdots + ax^2 + bx + c$, which has exactly n (not necessarily distinct) integral roots?

> **Walkthrough** —
>
> **(a)**

Solution 4. Assume that there exist nonzero reals a, b, c satisfying the given condition. Let $n > 3$ be a positive integer, and let $P_n(x) = x^n+\cdots+ax^2+bx+c$ be a polynomial with exactly n integral roots, which we denote by $r_1, r_2, \ldots, r_n$, counted with multiplicities. Since c is nonzero, none of the roots $r_1, r_2, \ldots, r_n$ is equal to zero. It follows that

$$\frac{1}{r_1}, \frac{1}{r_2}, \ldots, \frac{1}{r_n}$$

are the roots of the polynomial

$$Q_n(x) = x^n P\left(\frac{1}{x}\right) = cx^n + bx^{n-1} + ax^{n-2} + \cdots + x + 1.$$

By Vieta's formulas, we obtain

$$\frac{1}{r_1} + \cdots + \frac{1}{r_n} = -\frac{b}{c}, \quad \sum_{1 \leq i < j \leq n} \frac{1}{r_i r_j} = \frac{a}{c}.$$

This implies that

$$\left(\frac{1}{r_1} + \cdots + \frac{1}{r_n}\right)^2 - 2 \sum_{1 \leq i < j \leq n} \frac{1}{r_i r_j} = \frac{b^2 - 2ac}{c^2},$$

and hence

$$\frac{1}{r_1^2} + \cdots + \frac{1}{r_n^2} = \frac{b^2 - 2ac}{c^2}.$$

Also note that

$$r_1 r_2 \dots r_n = (-1)^n c.$$

It follows that

$$b^2 - 2ac \geq \frac{r_1^2 r_2^2 \dots r_n^2}{r_1^2} + \cdots + \frac{r_1^2 r_2^2 \dots r_n^2}{r_n^2} \geq n.$$

This shows that $n \leq b^2 - 2ac$ for any integer $n > 3$, which is a contradiction.

Therefore, there do not exist nonzero reals a, b, c satisfying the given condition. ■

Exercise 1.5 (All-Russian Mathematical Olympiad 2007 Grade 9 Day 1 P1, AoPS, by S. Berlov)**.** Let $f(x)$, $g(x)$ be monic quadratic polynomials with real coefficients such that the equations $f(g(x)) = 0$ and $g(f(x)) = 0$ have no real roots. Prove that at least one of the equations $f(f(x)) = 0$ and $g(g(x)) = 0$ has no real roots.

Walkthrough —

(a)

Solution 5. If one of the polynomials $f(x)$, $g(x)$ has no real roots, then the conclusion follows immediately. Thus, we may assume that both $f(x)$ and $g(x)$ have real roots. Denote the roots of $f(x)$ by α_1, α_2, and the roots of $g(x)$ by β_1, β_2. We need to show that none of α_1, α_2 lies in $f(\mathbb{R})$, or none of β_1, β_2 lies in $g(\mathbb{R})$.

Claim — If $\alpha_2 \leq \beta_2$, then none of α_1, α_2 lies in $f(\mathbb{R})$.

Proof of the Claim. Let us assume that $\alpha_1 \leq \beta_2$. If some of α_1, α_2 lies in $f(\mathbb{R})$, then noting that f is a monic quadratic polynomial, it follows from the intermediate value theorem that β_2 lies in $f(\mathbb{R})$, which shows that the equation $g(f(x)) = 0$ has a real root, a contradiction. This proves the claim. □

If $\alpha_2 \leq \beta_2$, then by the Claim, the equation $f(f(x)) = 0$ has no real roots. Similarly, if $\beta_2 \leq \alpha_2$, then the equation $g(g(x)) = 0$ has no real roots. This completes the proof. ■

Exercise 1.6 (All-Russian Mathematical Olympiad 2013 Grade 9 Day 1 P1, AoPS, by I. Bogdanov)**.** Given three distinct real numbers a, b, and c, show that at least two of the three following equations

$$\begin{aligned}(x-a)(x-b) &= x-c,\\ (x-b)(x-c) &= x-a,\\ (x-c)(x-a) &= x-b\end{aligned}$$

have real solutions.

Walkthrough —

(a)

Solution 6. Without loss of generality, let us consider the case that the equations

$$(x-a)(x-b) = x-c, (x-b)(x-c) = x-a$$

do not any real root. It follows that their discriminants are negative, that is,

$$(a+b+1)^2 - 4(ab+c), (b+c+1)^2 - 4(bc+a)$$

are negative. Note that

$$\begin{aligned}(a+b+1)^2 - 4(ab+c) &= (a-b)^2 + 1 + 2(a+b-2c)\\ &= (b-a+1)^2 + 4(a-c),\\ (b+c+1)^2 - 4(bc+a) &= (b-c)^2 + 1 + 2(b+c-2a)\\ &= (b-c+1)^2 + 4(c-a).\end{aligned}$$

Since a and c are distinct real numbers, one of $a-c$ and $c-a$ is positive. Thus, the discriminant of one of the equations

$$(x-a)(x-b) = x-c, (x-b)(x-c) = x-a$$

is non-negative, which is a contradiction. Therefore, at least two of the three equations have real solutions. ■

Solution 7. Let $f(x)$, $g(x)$, and $h(x)$ denote the polynomials defined by

$$\begin{aligned}f(x) &= (x-a)(x-b) - (x-c),\\ g(x) &= (x-b)(x-c) - (x-a),\end{aligned}$$

$$h(x) = (x-c)(x-a) - (x-b).$$

Note that

$$\begin{aligned} f(x) + g(x) &= (x-b)(2x-a-c) - (2x-a-c) \\ &= (2x-a-c)(x-b-1). \end{aligned}$$

This implies that $f(x) + g(x)$ admits a real root. Hence, at least one of $f(x)$ and $g(x)$ is non-positive at some real number. If one of them admits a real root, then we are done. Otherwise, at least one of them is negative at some real number. Also note that if k is a real number satisfying $k \geq |a| + |b| + |c| + 2$, then

$$\begin{aligned} f(k + |a| + |b|) &= (k + |a| + |b| + a)(k + |a| + |b| + b) - (k + |a| + |b| + c) \\ &\geq k^2 - k - |a| - |b| - |c| \\ &\geq k - |a| - |b| - |c| \\ &> 0 \end{aligned}$$

holds, and similarly,

$$g(k + |a| + |b|) > 0$$

holds too. By the intermediate value theorem, at least one of $f(x)$ and $g(x)$ admits a real root. Interchanging the roles of f, g, and h, it follows that at least one of any two of the polynomials f, g, and h admits a real root. Therefore, at least two of the three equations have real solutions for x. ■

Exercise 1.7 (All-Russian Mathematical Olympiad 2007 Grade 8 Day 1 P1, AoPS, by O. Podlipsky)**.** Given reals numbers a, b, c, prove that at least one of three equations

$$\begin{aligned} x^2 + (a-b)x + (b-c) &= 0, \\ x^2 + (b-c)x + (c-a) &= 0, \\ x^2 + (c-a)x + (a-b) &= 0 \end{aligned}$$

has a real root.

Walkthrough —

(a)

Solution 8. Note that the sum of the discriminants of the quadratic polynomials

$$x^2 + (a-b)x + (b-c), x^2 + (b-c)x + (c-a), x^2 + (c-a)x + (a-b)$$

are equal to

$$\begin{aligned}&(a-b)^2 - 4(b-c) + (b-c)^2 - 4(c-a) + (c-a)^2 - 4(a-b)\\&= (a-b)^2 + (b-c)^2 + (c-a)^2,\end{aligned}$$

which is non-negative, and hence, at least one of the discriminants is non-negative. This implies that at least one of the three equations has a real root. ■

Solution 9. Consider the three quadratic polynomials defined by

$$\begin{aligned}f(x) &= x^2 + (a-b)x + (b-c),\\g(x) &= x^2 + (b-c)x + (c-a),\\h(x) &= x^2 + (c-a)x + (a-b).\end{aligned}$$

Note that

$$f(0) + g(0) + h(0) = 0,$$

which implies that at least one of $f(0)$, $g(0)$, $h(0)$ is non-positive. Without loss of generality, assume that $f(0) \leq 0$. If $f(0)$ is zero, then $x = 0$ is a root of $f(x)$. If $f(0) < 0$, then noting that

$$\begin{aligned}&f(2|a-b| + 2|b-c| + 1)\\&= \frac{1}{2}(2|a-b| + 2|b-c| + 1)^2 + (a-b)(2|a-b| + 2|b-c| + 1)\\&\quad + \frac{1}{2}(2|a-b| + 2|b-c| + 1)^2 + (b-c)\\&= \frac{1}{2}(2|a-b| + 2|b-c| + 1)\left(2|a-b| + 2|b-c| + 1 - 2(a-b)\right)\\&\quad + \frac{1}{2}(2|a-b| + 2|b-c| + 1)^2 + (b-c)\\&\geq 1\end{aligned}$$

holds, it follows from the intermediate value theorem that $f(x)$ has a real root. ■

References

[**Che25**] Evan Chen. *The OTIS Excerpts*. Available at https://web.evanchen.cc/excerpts.html. 2025, pp. vi+289 (cited p. 1)