MOPSS

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Mathematics Olympiad

Problem Solving Sessions

MOPSS

Department of Mathematics

IISER Bhopal

https://jpsaha.github.io/MOTP/MOPSS/

Suggested readings

- Evan Chen's advice On reading solutions, available at https://blog.evanchen.cc/2017/03/06/on-reading-solutions/.
- Evan Chen's Advice for writing proofs/Remarks on English, available at https://web.evanchen.cc/handouts/english/english.pdf.
- Notes on proofs by Evan Chen from OTIS Excerpts [**Che25**, Chapter 1].
- Tips for writing up solutions by Edward Barbeau, available at https://www.math.utoronto.ca/barbeau/writingup.pdf.
- Evan Chen discusses why math olympiads are a valuable experience for high schoolers in the post on Lessons from math olympiads, available at https://blog.evanchen.cc/2018/01/05/lessons-from-math-olympiads/.

List of problems and examples

§1

Exercise 1.1 (All-Russian Mathematical Olympiad 2018 Grade 11 Day 1 P2, AoPS, by F. Petrov)**.** Let $n \geq 2$ and $x_1, x_2, \ldots, x_n$ be positive real numbers. Prove that

$$\frac{1+x_1^2}{1+x_1x_2} + \frac{1+x_2^2}{1+x_2x_3} + \cdots + \frac{1+x_n^2}{1+x_nx_1} \geq n.$$

Walkthrough —

(a) Use the Cauchy–Schwarz inequality to obtain a lower bound for each term.

(b) Use the AM–GM inequality to complete the proof.

Solution 1. Note that

$$(1+a^2)(1+b^2) \geq (1+ab)^2$$

holds for any real numbers a, b by the Cauchy–Schwarz inequality. This implies that

$$\begin{aligned}
&\frac{1+x_1^2}{1+x_1x_2} + \frac{1+x_2^2}{1+x_2x_3} + \cdots + \frac{1+x_n^2}{1+x_nx_1} \\
&\geq \sqrt{\frac{1+x_1^2}{1+x_2^2}} + \sqrt{\frac{1+x_2^2}{1+x_3^2}} + \cdots + \sqrt{\frac{1+x_n^2}{1+x_1^2}} \\
&\geq n,
\end{aligned}$$

where the last inequality follows from the AM–GM inequality. This completes the proof. ■

Exercise 1.2 (All-Russian Mathematical Olympiad 2013 Grade 9 Day 1 P1, AoPS)**.** Given three distinct real numbers a, b, and c, show that at least two of the three following equations

$$(x-a)(x-b) = x-c,$$

$$(x-b)(x-c) = x-a,$$
$$(x-c)(x-a) = x-b$$

have real solutions.

Walkthrough —

(a)

Solution 2. Without loss of generality, let us consider the case that the equations

$$(x-a)(x-b) = x-c, (x-b)(x-c) = x-a$$

do not any real root. It follows that their discriminants are negative, that is,

$$(a+b+1)^2 - 4(ab+c), (b+c+1)^2 - 4(bc+a)$$

are negative. Note that

$$\begin{aligned}(a+b+1)^2 - 4(ab+c) &= (a-b)^2 + 1 + 2(a+b-2c) \\ &= (b-a+1)^2 + 4(a-c), \\ (b+c+1)^2 - 4(bc+a) &= (b-c)^2 + 1 + 2(b+c-2a) \\ &= (b-c+1)^2 + 4(c-a).\end{aligned}$$

Since a and c are distinct real numbers, one of $a-c$ and $c-a$ is positive. Thus, the discriminant of one of the equations

$$(x-a)(x-b) = x-c, (x-b)(x-c) = x-a$$

is non-negative, which is a contradiction. Therefore, at least two of the three equations have real solutions. ■

Solution 3. Let $f(x)$, $g(x)$, and $h(x)$ denote the polynomials defined by

$$\begin{aligned}f(x) &= (x-a)(x-b) - (x-c), \\ g(x) &= (x-b)(x-c) - (x-a), \\ h(x) &= (x-c)(x-a) - (x-b).\end{aligned}$$

Note that

$$\begin{aligned}f(x) + g(x) &= (x-b)(2x-a-c) - (2x-a-c) \\ &= (2x-a-c)(x-b-1).\end{aligned}$$

This implies that $f(x) + g(x)$ admits a real root. Hence, at least one of $f(x)$ and $g(x)$ is non-positive at some real number. If one of them admits a real root, then we are done. Otherwise, at least one of them is negative at some real number. Also note that if k is a real number satisfying $k \geq |a| + |b| + |c| + 2$, then

$$\begin{aligned} f(k + |a| + |b|) &= (k + |a| + |b| + a)(k + |a| + |b| + b) - (k + |a| + |b| + c) \\ &\geq k^2 - k - |a| - |b| - |c| \\ &\geq k - |a| - |b| - |c| \\ &> 0 \end{aligned}$$

holds, and similarly,

$$g(k + |a| + |b|) > 0$$

hold too. By the intermediate value theorem, at least one of $f(x)$ and $g(x)$ admits a real root. Interchanging the roles of f, g, and h, it follows that at least one of any two of the polynomials f, g, and h admits a real root. Therefore, at least two of the three equations have real solutions for x. ■

Exercise 1.3 (All-Russian Mathematical Olympiad 1999 Grade 9 Day 1 P1, AoPS, by S. Volchenkov)**.** The decimal digits of a natural number A form an increasing sequence (from left to right). Find the sum of the digits of $9A$.

Walkthrough —

(a)

Solution 4.

■

Exercise 1.4 (All-Russian Mathematical Olympiad 2007 Grade 8 Day 1 P1, AoPS, by O. Podlipsky)**.** Given reals numbers a, b, c, prove that at least one of three equations

$$\begin{aligned} x^2 + (a - b)x + (b - c) &= 0, \\ x^2 + (b - c)x + (c - a) &= 0, \\ x^2 + (c - a)x + (a - b) &= 0 \end{aligned}$$

has a real root.

Walkthrough —

(a)

Solution 5. Note that the sum of the discriminants of the quadratic polynomials

$$x^2 + (a - b)x + (b - c), x^2 + (b - c)x + (c - a), x^2 + (c - a)x + (a - b)$$

are equal to

$$\begin{aligned}&(a-b)^2-4(b-c)+(b-c)^2-4(c-a)+(c-a)^2-4(a-b)\\&=(a-b)^2+(b-c)^2+(c-a)^2,\end{aligned}$$

which is non-negative, and hence, at least one of the discriminants is non-negative. This implies that at least one of the three equations has a real root. ■

Solution 6. Consider the three quadratic polynomials defined by

$$\begin{aligned}f(x)&=x^2+(a-b)x+(b-c),\\g(x)&=x^2+(b-c)x+(c-a),\\h(x)&=x^2+(c-a)x+(a-b).\end{aligned}$$

Note that

$$f(0)+g(0)+h(0)=0,$$

which implies that at least one of $f(0)$, $g(0)$, $h(0)$ is non-positive. Without loss of generality, assume that $f(0)\le 0$. If $f(0)$ is zero, then $x=0$ is a root of $f(x)$. If $f(0)<0$, then noting that

$$\begin{aligned}&f(2|a-b|+2|b-c|+1)\\&=\frac{1}{2}(2|a-b|+2|b-c|+1)^2+(a-b)(2|a-b|+2|b-c|+1)\\&\quad+\frac{1}{2}(2|a-b|+2|b-c|+1)^2+(b-c)\\&=\frac{1}{2}(2|a-b|+2|b-c|+1)\left(2|a-b|+2|b-c|+1-2(a-b)\right)\\&\quad+\frac{1}{2}(2|a-b|+2|b-c|+1)^2+(b-c)\\&\ge 1\end{aligned}$$

holds, it follows by the intermediate value theorem that $f(x)$ has a real root. ■

References

[**Che25**] Evan Chen. *The OTIS Excerpts*. Available at https://web.evanchen.cc/excerpts.html. 2025, pp. vi+289 (cited p. 1)