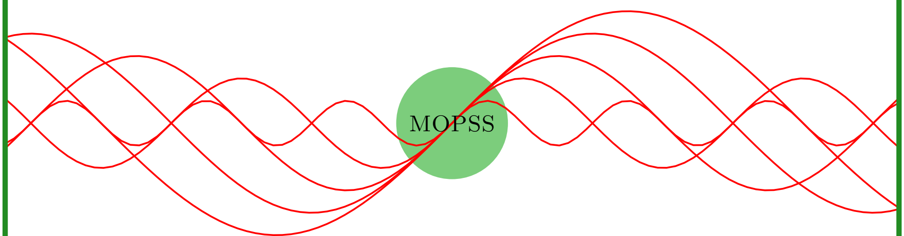
MOPSS

13 December 2025

Mathematics Olympiad

Problem Solving Sessions



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https://jpsaha.github.io/MOTP/MOPSS/

Suggested readings

- Evan Chen's advice On reading solutions, available at https://blog.evanchen.cc/2017/03/06/on-reading-solutions/.
- Evan Chen's Advice for writing proofs/Remarks on English, available at https://web.evanchen.cc/handouts/english/english.pdf.
- Notes on proofs by Evan Chen from OTIS Excerpts [**Che25**, Chapter 1].
- Tips for writing up solutions by Edward Barbeau, available at https://www.math.utoronto.ca/barbeau/writingup.pdf.
- Evan Chen discusses why math olympiads are a valuable experience for high schoolers in the post on Lessons from math olympiads, available at https://blog.evanchen.cc/2018/01/05/lessons-from-math-olympiads/.

List of problems and examples

§1

Exercise 1.1 (Japan Mathematical Olympiad 1992 P1)**.** Let x, y be coprime positive integers with $xy > 1$, and let n be an even positive integer. Prove that $x^n + y^n$ is not divisible by $x + y$.

Walkthrough —

(a)

Solution 1. On the contrary, let us assume that $x+y$ divides x^n+y^n. Since x is congruent to $-y$ modulo $x+y$, and n is even, it follows that x^n is congruent to y^n modulo $x+y$. This shows that x^n+y^n is congruent to $2y^n$ modulo $x+y$. Therefore, $x+y$ divides $2y^n$. Since x, y are coprime, it follows that $x+y$ and y^n are coprime. Hence, $x+y$ divides 2. Since x, y are positive integers with $xy > 1$, it follows that $x+y \geq 3$, which is a contradiction. Thus, $x+y$ does not divide x^n+y^n. ■

Exercise 1.2 (Japan Mathematical Olympiad 1993 P2)**.** For a positive integer n, let $d(n)$ denote its largest odd divisor. Define

$$\begin{aligned} D(n) &= d(1) + d(2) + \cdots + d(n), \\ T(n) &= 1 + 2 + \cdots + n. \end{aligned}$$

Show that there exist infinitely many positive integers n such that $3D(n) = 2T(n)$.

Walkthrough —

(a) Check that $n = 2, 6, 14, 30$ works.

(b) This may suggest that $n = 2^m - 2$ works for all $m \geq 2$.

(c) Does induction help?

Solution 2. Let us establish the following claim.

Claim — For any positive integer n,

$$D(2^n) = 2^n + \frac{1}{3}(2^n - 1)(2^n - 2)$$

holds.

Proof of the Claim. We prove the claim by induction on n. Note that

$$\begin{aligned} D(2) &= d(1) + d(2) \\ &= 1 + 1, \\ &= 2 + \frac{1}{3}(2-1)(2-2), \\ D(2^2) &= D(2) + d(3) + d(4) \\ &= 2 + 3 + 1 \\ &= 6 \\ &= 2^2 + \frac{1}{3}(2^2-1)(2^2-2) \end{aligned}$$

hold. Now, let $k \geq 2$ be a positive integer, and assume that the claim holds for for all positive integers up to k. Note that

$$\begin{aligned} & D(2^{k+1}) \\ &= D(2^k) + \sum_{i=1}^{2^k} d(2^k + i) \\ &= D(2^k) + \sum_{i=1}^{2^{k-1}} d(2^k + 2i - 1) + \sum_{i=1}^{2^{k-1}} d(2^k + 2i) \\ &= D(2^k) + \sum_{i=1}^{2^{k-1}} (2^k + 2i - 1) + \sum_{i=1}^{2^{k-1}} d(2^{k-1} + i) \\ &= D(2^k) + \frac{2^{k-1}}{2} \cdot (2^k + 1 + 2^k + 2 \cdot 2^{k-1} - 1) + D(2^k) - D(2^{k-1}) \\ &= 2D(2^k) - D(2^{k-1}) + 2^{k-2}(2^k + 2^{k+1}) \\ &= 2\left(2^k + \frac{1}{3}(2^k-1)(2^k-2)\right) \\ &\quad - \left(2^{k-1} + \frac{1}{3}(2^{k-1}-1)(2^{k-1}-2)\right) + 2^{k-2}(2^k + 2^{k+1}) \\ &= 2^{k+1} + \frac{1}{3}\left(2(2^k-1)(2^k-2) - (2^{k-1}-1)(2^{k-1}-2)\right) \\ &\quad + 2^{k-2}(2^k + 2^{k+1}) - 2^{k-1} \\ &= 2^{k+1} + \frac{1}{3}\left(2(2x-1)(2x-2) - (x-1)(x-2) + \frac{3}{2}x(2x+4x)\right) - x \\ &\quad (\text{where } x = 2^{k-1}) \\ &= 2^{k+1} + \frac{1}{3}\left(8x^2 - 12x + 4 - x^2 + 3x - 2 + 9x^2\right) - x \end{aligned}$$

$$
\begin{aligned}
&= 2^{k+1} + \frac{1}{3}(16x^2 - 12x + 2) \\
&= 2^{k+1} + \frac{1}{3}(4x - 1)(4x - 2) \\
&= 2^{k+1} + \frac{1}{3}(2^{k+1} - 1)(2^{k+1} - 2).
\end{aligned}
$$

□

For any integer $m \geq 2$, using the claim, we have

$$
\begin{aligned}
D(2^m - 2) &= D(2^m) - d(2^m - 1) - d(2^m) \\
&= D(2^m) - (2^m - 1) - 1 \\
&= 2^m + \frac{1}{3}(2^m - 1)(2^m - 2) - 2^m \\
&= \frac{1}{3}(2^m - 1)(2^m - 2),
\end{aligned}
$$

and consequently, we obtain

$$
\begin{aligned}
3D(2^m - 2) &= (2^m - 1)(2^m - 2) \\
&= 2(1 + 2 + \cdots + (2^m - 2)) \\
&= 2T(2^m - 2).
\end{aligned}
$$

■

References

[**Che25**] Evan Chen. *The OTIS Excerpts*. Available at https://web.evanchen.cc/excerpts.html. 2025, pp. vi+289 (cited p. 1)