IOQM 2023

MOPSS

17 June 2024

Mathematics Olympiad

Problem Solving Sessions

MOPSS

Department of Mathematics
IISER Bhopal

https://jpsaha.github.io/MOTP/MOPSS/

Suggested readings

- Evan Chen's
  - advice *On reading solutions*, available at https://blog.evanchen.cc/2017/03/06/on-reading-solutions/.
  - *Advice for writing proofs/Remarks on English*, available at https://web.evanchen.cc/handouts/english/english.pdf.
- Evan Chen discusses why *math olympiads are a valuable experience for high schoolers* in the post on *Lessons from math olympiads*, available at https://blog.evanchen.cc/2018/01/05/lessons-from-math-olympiads/.

List of problems and examples

Question 1

Let n be a positive integer such that $1 \leq n \leq 1000$. Let M_n be the number of integers in the set

$$X_n = \{\sqrt{4n+1}, \sqrt{4n+2}, \dots, \sqrt{4n+1000}\}.$$

Let

$$a = \max\{M_n \,|\, 1 \leq n \leq 1000\}, \text{ and } b = \min\{M_n \,|\, 1 \leq n \leq 1000\}.$$

Find $a - b$.

Answer: 22.

Summary — Note that the gap between two consecutive squares increases as one **moves to the right** of the number line (because the difference between $(n+1)^2$ and n^2 is a linear polynomial in n with positive coefficients).

Walkthrough —

(a) Given a sequence of consecutive 1000 positive integers, we need to understand how many of them are perfect squares.

(b) Note that the gap between two consecutive squares increases as one **moves to the right**.

(c) Draw the number line to convince yourself that

$$M_1 \geq M_2 \geq M_2 \geq \cdots \geq M_{1000}.$$

(d) Note that $31^2 < 1000 < 32^2$, and hence

$$a = M_1 = 31 - 2 = 29.$$

(e) Also note that $63^2 < 4001 < 64^2$, and $70^2 < 5000 < 71^2$, and hence

$$b = M_{1000} = 70 - 63 = 7.$$

(f) Conclude that $a - b = 22$.

Question 2

Find the number of elements in the set

$$\{(a, b) \in \mathbb{N} \times \mathbb{N} \mid 2 \leq a, b \leq 2023, \log_a(b) + 6\log_b(a) = 5\}.$$

Answer: 54.

Summary — Quadratic equation!

Walkthrough —

(a) Note that the condition

$$\log_a(b) + 6\log_b(a) = 5$$

is equivalent to

$$\log_a(b) = 2, \text{ or } 3,$$

which is equivalent to $b = a^2$ or $b = a^3$.

(b) This shows that the given set is equal to the union of the sets

$$\{(a, b) \in \mathbb{N} \times \mathbb{N} \mid 2 \leq a, b \leq 2023, b = a^2\}$$

and

$$\{(a, b) \in \mathbb{N} \times \mathbb{N} \mid 2 \leq a, b \leq 2023, b = a^3\}.$$

(c) Observe that

$$44^2 < 2023 < 45^2, 12^3 = 144 \times 12 < 2023, 13^3 = 169 \times 13 > 2023.$$

(d) Note that the above sets have no common elements. It follows that the number of elements of the given set is $44 - 1 + 12 - 1 = 54$.

Question 3

Let α, β be positive integers such that

$$\frac{16}{37} < \frac{\alpha}{\beta} < \frac{7}{16}.$$

Find the smallest possible value of β.

Question 4

Let x, y be positive integer such that

$$x^4 = (x-1)(y^3 - 23) - 1.$$

Find the maximum possible value of $x + y$.

Answer: 7.

Summary — Look for some divisibility conditions.

Walkthrough —

(a) Note that $x \neq 1$ and $x - 1$ divides 2. It follows that $x = 2$ or $x = 3$.

(b) If $x = 2$, then $y^3 = 23 + 2^4 + 1 = 40$, which is impossible since y is an integer.

(c) If $x = 3$, then $y^3 = 23 + \frac{1}{2}\left(3^4 + 1\right) = 23 + 41 = 64$, which gives $y = 4$.

(d) It follows that $x + y = 7$.

Question 5

In a triangle ABC, let E be the midpoint of AC and F be the midpoint of AB. The medians BE and CF intersect at G. Let Y and Z be the midpoints of BE and CF respectively. If the area of the triangle ABC is 480, find the area of the triangle GYZ.

Answer: 10.

Summary — The ratio of the areas of two triangles with equal heights equal to the ratio of their bases. One often says that the area of a triangle with constant height is proportional to its base.

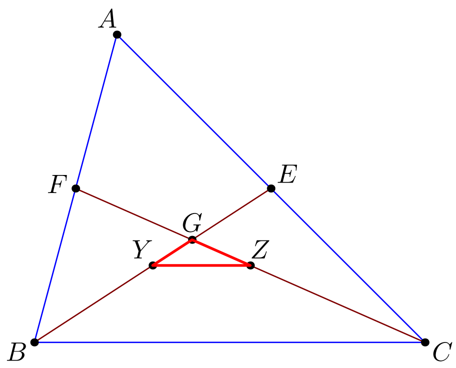

Figure 1: IOQM23 Q5, Question 5

Walkthrough —

(a) The area of the triangle BCF is equal to 240.

(b) The area of the triangle BCG is equal to $\frac{2}{3} \times 240 = 160$.

(c) Note that $BY : YG : GE$ is equal to $3 : 1 : 2$. This shows that $BY : YG = 3 : 1$. Similarly, $CZ : ZG = 3 : 1$.

(d) It follows that YZ is parallel to BC, and $YZ : BC = 1 : 4$.

(e) The ratio of the height of GYZ to the height of BCG is equal to $1 : 4$.

(f) Consequently, the area of GYZ is equal to $\frac{1}{4^2} \times 160 = 10$.

Question 6

Let X be the set of all even positive integers n such that the measure of the angle subtended by a side at the center of some regular polygon is n degrees. Find the number of elements in X.

Answer: 16.

Walkthrough —

(a) Let $r \geq 3$ be an integer and consider a regular r-gon. Note that $360/r$ is even if and only if r is equal to one of

$$3, 4, 5, 6, 9, 10, 12, 15, 18, 20, 24, 36, 45, 60, 120, 360.$$

In other words, there are precisely 16 positive integers $r \geq 3$ such that dividing 360 by r yields an even number.

(b) It follows that the number of elements of X is equal to 16.

Question 7

Unconventional dice are to be designed such that the six faces are marked with numbers from 1 to 6 with 1 and 2 appearing on opposite faces. Further, each face is colored either red or yellow with opposite faces always of the same color. Two dice are considered to have the same design if one of them can be rotated to obtain a dice that has the same numbers and colors on the corresponding faces as the other one. Find the number of distinct dice that can be designed.

Answer: 48.

Walkthrough —

(a) Fix a pair of opposite faces (say, top and the bottom one, for ease of reference), which are to be marked using 1 and 2.

(b) Note that after marking the top and bottom face using 1 and 2 (in some order), the remaining faces are to be marked using $3, 4, 5, 6$, and thus, the marking of the remaining faces correspond to the circular permutations of $3, 4, 5, 6$.

(c) This shows that the marking can be done in 6 ways.

(d) Note that there are precisely 6 dice of the prescribed type having the same marking.

(e) Hence, the number of distinct dice is equal to $6 \times 2^3 = 48$.

Question 8

Given a 2×2 tile and seven dominoes (2×1 tile), find the number of ways of tiling (that is, cover without leaving gaps and without overlapping of any two tiles) a 2×7 rectangle using some of these tiles.

Question 9

Find the number of triples (a, b, c) of positive integers such that

(a) ab is a prime,

(b) bc is a product of two primes,

(c) abc is not divisible by square of any prime and

(d) $abc \leq 30$.

Answer: 17.

Walkthrough —

(a) Suppose (a, b, c) is a pair satisfying the above conditions.

(b) Since ab is a prime, it follows that one of a, b is equal to 1, and the remaining one is a prime.

(c) If $a = 1$, then b, c are distinct primes.

(d) If $b = 1$, then c is the product of two distinct primes which are not equal to a.

(e) This shows that (a, b, c) is equal to one of

$$\begin{aligned}
&(1,2,3),(1,2,5),(1,2,7),(1,2,11),(1,2,13),\\
&(1,3,2),(1,3,5),(1,3,7),\\
&(1,5,2),(1,5,3),\\
&(1,7,2),(1,7,3),\\
&(1,11,2),\\
&(1,13,2),\\
&(2,1,3\cdot 5),(3,1,2\cdot 5),(5,1,2\cdot 3).
\end{aligned}$$

(f) Hence, the number of triples satisfying the above conditions is at most 16.

(g) Since any of the above triples does satisfy the given conditions, it follows that the number of triples satisfying the given conditions is equal to 17.

Question 10

The sequence $\langle a_n \rangle_{n \geq 0}$ is defined by $a_0 = 1, a_1 = -4$ and

$$a_{n+2} = -4a_{n+1} - 7a_n$$

for $n \geq 0$. Find the number of positive integer divisors of $a_{50}^2 - a_{49}a_{51}$.

Answer: 51.

Walkthrough —

(a) Compute the first few terms of the sequence to note that

$$\begin{aligned}
a_0 &= 1,\\
a_1 &= -4,\\
a_2 &= 9,\\
a_3 &= -8,\\
a_4 &= -31,
\end{aligned}$$

which gives

$$\begin{aligned} a_1^2 - a_0a_2 &= 16 - 9 \\ &= 7, \\ a_2^2 - a_1a_3 &= 81 - 32 \\ &= 49, \\ a_3^2 - a_2a_4 &= 64 + 279 \\ &= 343. \end{aligned}$$

This may suggest that

$$a_n^2 - a_{n-1}a_{n+1} = 7^n$$

holds for any integer $n \geq 1$.

(b) Indeed, it is clear for $n = 1$. Assuming that

$$a_n^2 - a_{n-1}a_{n+1} = 7^n$$

holds for some integer $n \geq 1$, note that

$$\begin{aligned} a_{n+1}^2 - a_na_{n+2} &= a_{n+1}^2 - a_n(-4a_{n+1} - 7a_n) \\ &= a_{n+1}^2 + 4a_na_{n+1} + 7a_n^2 \\ &= a_{n+1}^2 + 4a_na_{n+1} + 7a_{n-1}a_{n+1} + 7^{n+1} \\ &= a_{n+1}(a_{n+1} + 4a_n + 7a_{n-1}) + 7^{n+1} \\ &= 7^{n+1}. \end{aligned}$$

This proves that

$$a_n^2 - a_{n-1}a_{n+1} = 7^n$$

holds for any integer $n \geq 1$.

(c) Consequently,

$$a_{50}^2 - a_{49}a_{51} = 7^{50},$$

whose number of positive integer divisors is equal to 51.

Question 11

A positive integer m has the property that m^2 is expressible in the form $4n^2 - 5n + 16$ where n is an integer (of any sign). Find the maximum possible value of $|m - n|$.

Answer: 14.

Walkthrough —

(a) Let m, n be integers satisfying $m^2 = 4n^2 - 5n + 16$, or equivalently,

$$4n^2 - 5n + (16 - m^2) = 0.$$

(b) Since n is an integer, the discriminant of the quadratic polynomial $4n^2 - 5n + (16 - m^2)$ in n is a perfect square, that is, there exists an integer ℓ such that

$$25 - 16(16 - m^2) = \ell^2,$$

or equivalently, $16m^2 - \ell^2 = 231$, which gives

$$(4m - \ell)(4m + \ell) = 3 \cdot 7 \cdot 11.$$

(c) Note that $4m - \ell, 4m + \ell$ are odd and hence congruent to $1, -1$ modulo 4 (in some order). Also note that $4m - \ell < 4m + \ell$ holds.

(d) This shows that $(4m - \ell, 4m + \ell)$ is equal to one of

$$(1, 3 \cdot 7 \cdot 11), (3, 77), (7, 33), (11, 21),$$

and consequently, $8m = 232, 80, 40, 32$, which yields $m = 29, 10, 5, 4$.

(e) If $m = 29$, then $4n^2 - 5n - 825 = 0$, which gives $n = 15$.

(f) If $m = 10$, then $4n^2 - 5n - 84 = 0$ holds, which gives $n = -4$.

(g) If $m = 5$, then $4n^2 - 5n - 9 = 0$ holds, implying $n = -1$.

(h) If $m = 4$, then $n = 0$.

(i) Conclude that the maximum possible value of $|m - n|$ is equal to 14.

Question 12

Let $P(x) = x^3 + ax^2 + bx + c$ be a polynomial where a, b, c are integers and c is odd. Let p_i be the value of $P(x)$ at $x = i$. Given that $p_1^3 + p_2^3 + p_3^3 = 3p_1p_2p_3$, find the value of $p_2 + 2p_1 - 3p_0$.

Answer: 18.

Walkthrough —

(a) Note that if $p_1 + p_2 + p_3 \neq 0$, then using the condition that p_1, p_2, p_3 are real, it follows that $p_1 = p_2 = p_3$.

(b) Note that

$$\begin{aligned} p_1 &= 1 + a + b + c, \\ p_2 &= 8 + 4a + 2b + c, \\ p_3 &= 27 + 9a + 3b + c, \end{aligned}$$

which gives that

$$p_1 + p_2 + p_3 = 36 + 14a + 6b + 3c.$$

Since c is an odd integer, it follows that $p_1 + p_2 + p_3 \neq 0$. Consequently, $p_1 = p_2 = p_3$.

(c) This gives

$$3a + b = -7, 5a + b = -19,$$

and hence, $a = -6, b = 11$.

(d) This gives

$$\begin{aligned} p_2 + 2p_1 - 3p_0 &= 10 + 6a + 4b + 3c - 3c \\ &= 10 - 36 + 44 \\ &= 18. \end{aligned}$$

Question 13

The ex-radii of a triangle are $10\frac{1}{2}$, 12 and 14. If the sides of the triangle are the roots of the cubic $x^3 - px^2 + qx - r = 0$, where p, q, r are integers, find the integer nearest to $\sqrt{p+q+r}$.

Answer: 58.

Walkthrough —

(a) Denote the sides of the triangle by a, b, c. Let s denote its perimeter, and Δ denote its area.

(b) Using

$$\Delta = (s-a)r_1 = (s-b)r_2 = (s-c)r_3,$$

we obtain

$$\Delta = \frac{21}{2}(s-a) = 12(s-b) = 14(s-c).$$

This gives

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{s\Delta^3 \times \frac{2}{21} \times \frac{1}{12} \times \frac{1}{14}} = \frac{1}{42}\Delta\sqrt{s\Delta},$$

implying that

$$s\Delta = 42^2.$$

(c) Also note that

$$s - a = \frac{2}{21}\Delta, s - b = \frac{1}{12}\Delta, s - c = \frac{1}{14}\Delta,$$

yields

$$s = \left(\frac{2}{21} + \frac{1}{12} + \frac{1}{14}\right)\Delta = \frac{8+7+6}{84}\Delta = \frac{1}{4}\Delta.$$

(d) It follows that

$$s = 21, \Delta = 84.$$

(e) Moreover, note that

$$\begin{aligned} a &= s - \frac{2}{21}\Delta \\ &= 21 - 8 \\ &= 13, \\ b &= s - \frac{1}{12}\Delta \\ &= 21 - 7 \\ &= 14, \\ c &= s - \frac{1}{14}\Delta \\ &= 21 - 6 \\ &= 15. \end{aligned}$$

(f) Since a, b, c are the roots of the cubic $x^3 - px^2 + qx - r = 0$, it follows that

$$-p - q - r - 1 = (-1-a)(-1-b)(-1-c) = -14 \cdot 15 \cdot 16,$$

which gives

$$p + q + r = 3360 - 1 = 3359.$$

(g) Check that the integer nearest to $\sqrt{3359}$ is 58.

Question 14

Let ABC be a triangle in the xy plane, where B is at the origin $(0,0)$. Let BC be produced to D such that $BC : CD = 1 : 1$, CA be produced to E such that $CA : AE = 1 : 2$ and AB be produced to F such that $AB : BF = 1 : 3$. Let $G(32, 24)$ be the centroid of the triangle ABC and K be the centroid of the triangle DEF. Find the length GK.

Answer: 40.

Walkthrough —

(a) Note that

$$\begin{aligned} D &= 2C - B, \\ E &= 3A - 2C, \\ F &= 4B - 3A, \end{aligned}$$

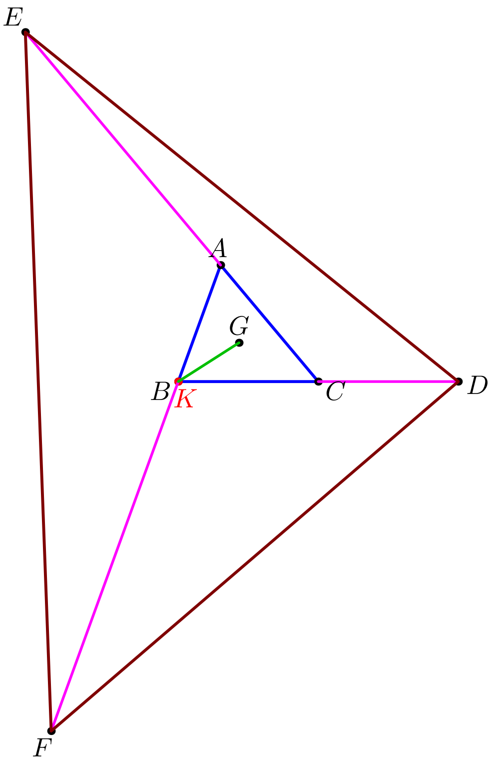

Figure 2: IOQM23 Q14, Question 14

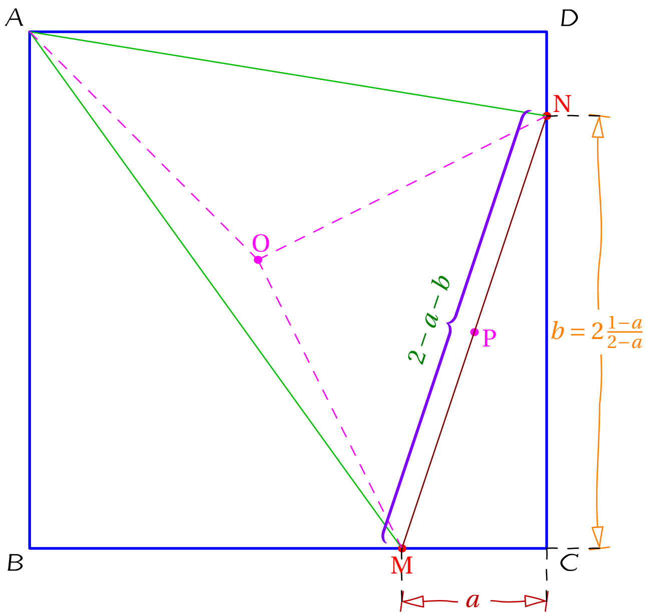

Figure 3: IOQM23 Q15, Question 15

which gives

$$K = \frac{D+E+F}{3} = B.$$

Since B is at the origin $(0,0)$ and G has coordinates $(32,24)$, it follows that

$$GK = \sqrt{32^2+24^2} = 8\cdot\sqrt{4^2+3^2} = 40.$$

Question 15

Let $ABCD$ be a unit square. Suppose M and N are points on BC and CD respectively such that the perimeter of triangle MCN is 2. Let O be the circumcentre of triangle MAN, and P be the circumcentre of triangle MON. If $\left(\frac{OP}{OA}\right)^2 = \frac{m}{n}$ for some relatively prime positive integers m and n, find the value of $m+n$.

Answer: 3.

Walkthrough —

(a) Choose a, b as in Fig. 3 such that

$$a^2+b^2 = (2-a-b)^2,$$

which gives

$$4 - 4a = (4 - 2a)b,$$

and hence

$$b = 2\frac{1-a}{2-a}$$

Note that

$$\begin{aligned}
AM^2 &= 1 + (1-a)^2 \\
&= 2 - 2a + a^2, \\
AN^2 &= 1 + (1-b)^2 \\
&= 2 - 2b + b^2 \\
&= 2\left(1 - 2\frac{1-a}{2-a} + 2\left(\frac{1-a}{2-a}\right)^2\right) \\
&= \frac{2}{(2-a)^2}\left(4 - 4a + a^2 - 4 + 6a - 2a^2 + 2 - 4a + 2a^2\right) \\
&= \frac{2}{(2-a)^2}\left(2 - 2a + a^2\right), \\
AM^2 + AN^2 - MN^2 &= 1 + (1-a)^2 + 1 + (1-b)^2 - (2-a-b)^2 \\
&= 2 + (1-a)^2 + (1-b)^2 - a^2 - b^2 \\
&= 4 - 2a - 2b \\
&= 4 - 2a - 4\frac{1-a}{2-a} \\
&= 2\left(2 - a - \frac{2-2a}{2-a}\right) \\
&= 2\frac{2-2a+a^2}{2-a} \\
&= \sqrt{2}AM \cdot AN.
\end{aligned}$$

This gives that $\angle MAN = 45^\circ$. Since O is sthe circumcenter of MAN, it follows that $\angle MON = 90^\circ$. Moreover, $OA = OM = ON$. This implies that MON is a right-angled isosceles triangle. Since P is the circumcenter of MON, it follows that OPN is a right-angled isosceles triangle, which gives $OP = \frac{1}{\sqrt{2}}ON$. This implies that $\left(\frac{OP}{OA}\right)^2 = \frac{1}{2}$, and hence $m + n = 3$.

Question 16

The six sides of a convex hexagon $A_1A_2A_3A_4A_5A_6$ are colored red. Each of the diagonals of the hexagon is colored either red or blue. If N is the number of such colorings such that every triangle $A_iA_jA_k$, where $1 \leq i < j < k \leq 6$, has at least one red side, find the sum of the squares of the digits of N.

Answer: 94.

Walkthrough —

(a) The sides of the hexagon has been colored red.

(b) Note that N is the number of colorings of the diagonals of the hexagon using red and blue such that each of the triangles $A_1A_2A_3$ and $A_2A_4A_6$ have at least one red side.

(c) The number of ways of coloring the edges of $A_1A_2A_3$ using red and blue such that $A_1A_2A_3$ has at least one red side is $2^3 - 1 = 7$.

(d) For each of these colorings, the number of ways of colorings of the edges of $A_2A_4A_6$ using red and blue such that $A_2A_4A_6$ has at least one red side is $2^3 - 1 = 7$.

(e) For any such colorings of the edges of $A_1A_2A_3$ and of the edges of $A_2A_4A_6$, the remaining $\binom{6}{2} - 6 - 3 - 3 = 3$ diagonals of the hexagon can be colored using red and blue using $2^3 = 8$ ways.

(f) It follows that $N = 7 \cdot 7 \cdot 8 = 392$.

(g) The sum of the squares of the digits of N is equal to $3^2 + 9^2 + 2^2 = 9 + 81 + 4 = 94$.