Invariance principle

MOPSS

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Mathematics Olympiad

Problem Solving Sessions

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https://jpsaha.github.io/MOTP/MOPSS/

Suggested readings

- Evan Chen's advice On reading solutions, available at https://blog.evanchen.cc/2017/03/06/on-reading-solutions/.
- Evan Chen's Advice for writing proofs/Remarks on English, available at https://web.evanchen.cc/handouts/english/english.pdf.
- Notes on proofs by Evan Chen from OTIS Excerpts [**Che25**, Chapter 1].
- Tips for writing up solutions by Edward Barbeau, available at https://www.math.utoronto.ca/barbeau/writingup.pdf.
- Evan Chen discusses why math olympiads are a valuable experience for high schoolers in the post on Lessons from math olympiads, available at https://blog.evanchen.cc/2018/01/05/lessons-from-math-olympiads/.

List of problems and examples

§1 Invariance principle

See [FGI96, Chapter 12], [Eng98, Chapter 1].

Example 1.1 (India RMO 2014b P6). [Eng98, Problem 28, p. 29, p. 35] Let n be an odd positive integer and suppose that each square of an $n \times n$ grid is arbitrarily filled with either by 1 or by -1. Let r_j and c_k denote the product of all numbers in j-th row and k-th column respectively, $1 \leq j, k \leq n$. Prove that

$$\sum_{j=1}^{n} r_j + \sum_{k=1}^{n} c_k \neq 0.$$

> **Walkthrough** —
>
> **(a)** What would happen to the sum if the sign of one of the entries is changed? Does the sum change modulo 4?
>
> **(b)** What would happen when all the entries are changed to 1 by changing the signs of the negative entries?

Solution 1. Denote the sum $\sum_{j=1}^{n} r_j + \sum_{k=1}^{n} c_k$ by S. If all entries of the grid are equal to 1, then S is equal to $2n$. Note that if the sign on an entry of the grid is changed, then S does not change modulo 4. So the integer S is congruent to $2n$ modulo 4. Since n is odd, the integer S is nonzero. For an alternate solution, see [Eng98, Problem 29, Chapter 2, p. 28]. ■

Example 1.2 (India RMO 2016d P4). A box contains 4032 answer scripts out of which exactly half have odd number of marks. We choose 2 scripts randomly and, if the scores on both of them are odd number, we add one mark to one of them, put the script back in the box and keep the other script outside. If both scripts have even scores, we put back one of the scripts and keep the other outside. If there is one script with even score and the other with odd score, we put back the script with the odd score and keep the other script outside. After following this procedure a number of times, there are 3 scripts left among which there is at least one script each with odd and even scores. Find, with proof, the number of scripts with odd scores among the three left.

> **Walkthrough** — What happens to the number of scripts with odd scores?

Solution 2. Note that under this process, the number of scripts with odd scores remains unchanged or decreases by two. Consequently, if this process is repeated few times, then the number of scripts with odd scores would remain unchanged or decrease by an even number. Note that the box contains $4032 \times \frac{1}{2} = 2016$ scripts with odd scores. So after following the process certain number of times, the number of scripts with odd number of marks will always be an even number. The number of scripts with odd scores among the three left being at least one, is equal two. ■

References

[**Che25**] Evan Chen. *The OTIS Excerpts*. Available at https://web.evanchen.cc/excerpts.html. 2025, pp. vi+289 (cited p. 1)

[**Eng98**] Arthur Engel. *Problem-solving strategies*. Problem Books in Mathematics. Springer-Verlag, New York, 1998, pp. x+403. isbn: 0-387-98219-1 (cited p. 2)

[**FGI96**] Dmitri Fomin, Sergey Genkin, and Ilia Itenberg. *Mathematical circles (Russian experience)*. Vol. 7. Mathematical World. Translated from the Russian and with a foreword by Mark Saul. American Mathematical Society, Providence, RI, 1996, pp. xii+272. isbn: 0-8218-0430-8 (cited p. 2)