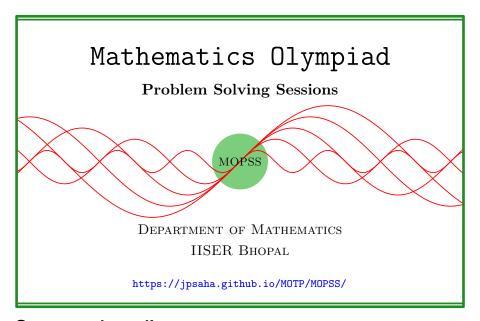
Generating functions

MOPSS

7 May 2025



Suggested readings

- Evan Chen's advice On reading solutions, available at https://blog.evanchen.cc/2017/03/06/on-reading-solutions/.
- Evan Chen's Advice for writing proofs/Remarks on English, available at https://web.evanchen.cc/handouts/english/english.pdf.
- Notes on proofs by Evan Chen from OTIS Excerpts [Che25, Chapter 1].
- Tips for writing up solutions by Edward Barbeau, available at https://www.math.utoronto.ca/barbeau/writingup.pdf.
- Evan Chen discusses why math olympiads are a valuable experience for high schoolers in the post on Lessons from math olympiads, available at https://blog.evanchen.cc/2018/01/05/lessons-from-math-olympiads/.

List of problems and examples

§1 Generating functions

See [Sob13, Chapter 6], [Wil06].

Example 1.1. In how many ways, can we fill a bag with n fruits subject to the following constraints?

- 1. The number of apples must be even.
- 2. The number of bananas must be a multiple of 5.
- 3. There can be at most four oranges.
- 4. There can be at most one pear.

Solution 1. Note that the required number is equal to the coefficient of x^n in the formal power series A(x)B(x)O(x)P(x) where

$$A(x) = 1 + x^{2} + x^{4} + x^{6} + \dots,$$

$$B(x) = 1 + x^{5} + x^{10} + x^{15} + \dots,$$

$$O(x) = 1 + x + x^{2} + x^{3} + x^{4},$$

$$P(x) = 1 + x.$$

Also note that

$$A(x)B(x)O(x)P(x) = \frac{1}{1-x^2} \frac{1}{1-x^5} \frac{1-x^5}{1-x} \times (1+x)$$
$$= \frac{1}{(1-x)^2}$$
$$= 1 + 2x + 3x^2 + 4x^3 + \dots$$

So the bag can be filled in n+1 ways such that the given conditions hold.

References

- [Che25] EVAN CHEN. The OTIS Excerpts. Available at https://web.evanchen.cc/excerpts.html. 2025, pp. vi+289 (cited p. 1)
- [Sob13] PABLO SOBERÓN. Problem-solving methods in combinatorics. An approach to olympiad problems. Birkhäuser/Springer Basel AG, Basel, 2013, pp. x+174. ISBN: 978-3-0348-0596-4; 978-3-0348-0597-1. DOI: 10.1007/978-3-0348-0597-1. URL: http://dx.doi.org/10.1007/978-3-0348-0597-1 (cited p. 2)

[Wil06] Herbert S. Wilf. generating functionology. Third. A K Peters, Ltd., Wellesley, MA, 2006, pp. x+245. ISBN: 978-1-56881-279-3; 1-56881-279-5 (cited p. 2)