Generating functions

MOPSS

7 May 2025

Mathematics Olympiad

Problem Solving Sessions

MOPSS

Department of Mathematics
IISER Bhopal

https://jpsaha.github.io/MOTP/MOPSS/

Suggested readings

- Evan Chen's advice On reading solutions, available at https://blog.evanchen.cc/2017/03/06/on-reading-solutions/.
- Evan Chen's Advice for writing proofs/Remarks on English, available at https://web.evanchen.cc/handouts/english/english.pdf.
- Notes on proofs by Evan Chen from OTIS Excerpts [**Che25**, Chapter 1].
- Tips for writing up solutions by Edward Barbeau, available at https://www.math.utoronto.ca/barbeau/writingup.pdf.
- Evan Chen discusses why math olympiads are a valuable experience for high schoolers in the post on Lessons from math olympiads, available at https://blog.evanchen.cc/2018/01/05/lessons-from-math-olympiads/.

List of problems and examples

1.1 Example 2

§1 Generating functions

See [**Sob13**, Chapter 6], [**Wil06**].

Example 1.1. In how many ways, can we fill a bag with n fruits subject to the following constraints?

1. The number of apples must be even.
2. The number of bananas must be a multiple of 5.
3. There can be at most four oranges.
4. There can be at most one pear.

Solution 1. Note that the required number is equal to the coefficient of x^n in the formal power series $A(x)B(x)O(x)P(x)$ where

$$\begin{aligned} A(x) &= 1 + x^2 + x^4 + x^6 + \dots, \\ B(x) &= 1 + x^5 + x^{10} + x^{15} + \dots, \\ O(x) &= 1 + x + x^2 + x^3 + x^4, \\ P(x) &= 1 + x. \end{aligned}$$

Also note that

$$\begin{aligned} A(x)B(x)O(x)P(x) &= \frac{1}{1-x^2}\frac{1}{1-x^5}\frac{1-x^5}{1-x} \times (1+x) \\ &= \frac{1}{(1-x)^2} \\ &= 1 + 2x + 3x^2 + 4x^3 + \dots. \end{aligned}$$

So the bag can be filled in $n+1$ ways such that the given conditions hold. ■

References

[**Che25**] Evan Chen. *The OTIS Excerpts*. Available at https://web.evanchen.cc/excerpts.html. 2025, pp. vi+289 (cited p. 1)

[**Sob13**] Pablo Soberón. *Problem-solving methods in combinatorics*. An approach to olympiad problems. Birkhäuser/Springer Basel AG, Basel, 2013, pp. x+174. isbn: 978-3-0348-0596-4; 978-3-0348-0597-1. doi: 10.1007/978-3-0348-0597-1. url: http://dx.doi.org/10.1007/978-3-0348-0597-1 (cited p. 2)

[**Wil06**] Herbert S. Wilf. *generatingfunctionology*. Third. A K Peters, Ltd., Wellesley, MA, 2006, pp. x+245. isbn: 978-1-56881-279-3; 1-56881-279-5 (cited p. 2)