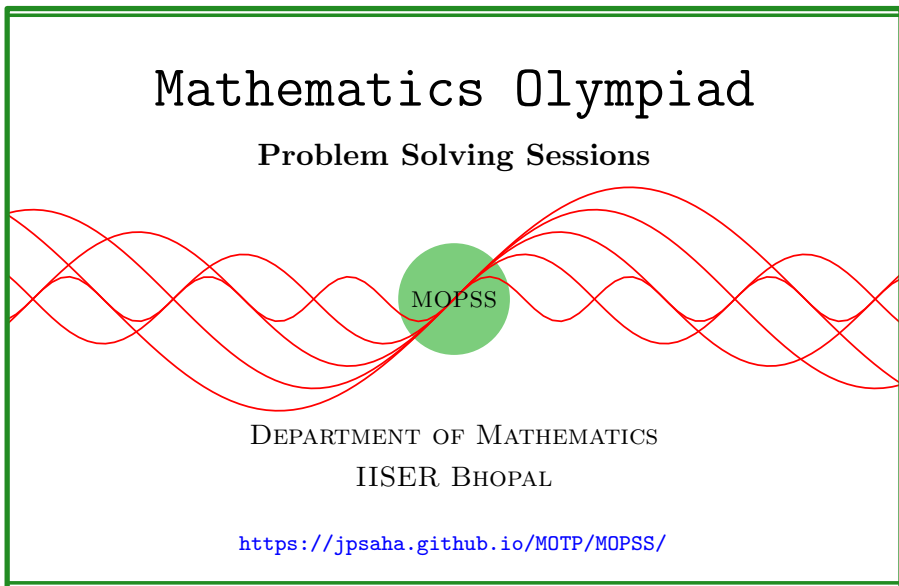


# Extremal principle

MOPSS

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## Suggested readings

- Evan Chen's advice [On reading solutions](https://blog.evanchen.cc/2017/03/06/on-reading-solutions/), available at <https://blog.evanchen.cc/2017/03/06/on-reading-solutions/>.
- Evan Chen's [Advice for writing proofs/Remarks on English](https://web.evanchen.cc/handouts/english/english.pdf), available at <https://web.evanchen.cc/handouts/english/english.pdf>.
- [Notes on proofs](#) by Evan Chen from [OTIS Excerpts](#) [[Che25](#), Chapter 1].
- [Tips for writing up solutions](https://www.math.utoronto.ca/barbeau/writingup.pdf) by Edward Barbeau, available at <https://www.math.utoronto.ca/barbeau/writingup.pdf>.
- Evan Chen discusses why [math olympiads are a valuable experience for high schoolers](#) in the post on [Lessons from math olympiads](#), available at <https://blog.evanchen.cc/2018/01/05/lessons-from-math-olympiads/>.

# List of problems and examples

1.1	Example (India RMO 1991 P8)	2
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## §1 Extremal principle

See [Eng98, Chapter 3].

**Example 1.1** (India RMO 1991 P8). The 64 squares of an  $8 \times 8$  chessboard are filled with positive integers in such a way that each integer is the average of the integers on the neighbouring squares. (Two squares are neighbours if they share a common edge or a common vertex. Thus a square can have 8, 5 or 3 neighbours depending on its position). Show that all the 64 integer entries are in fact equal.

### Walkthrough —

- (a) Consider a square containing the maximum (or minimum) of all the entries, denoted by  $m$ .
- (b) Show that the entries of the neighbouring squares are equal to  $m$ .
- (c) What are the possibilities for the entries of the remaining squares?

**Solution 1.** Let  $m$  denote the maximum among the entries of the 64 squares. Note that if a square contains  $m$ , then the entries of its neighbouring squares are equal to  $m$  (otherwise, the entry of some of the neighbouring squares would be strictly smaller than  $m$ , then the average of the entries of the neighbouring squares would be strictly smaller than  $m$ , which is impossible). Consequently, the entries of any two neighbouring squares are equal to  $m$  if one of them contains  $m$ . Let  $S$  denote a square containing  $m$ . Note that any other square on the chessboard can be reached from  $S$  through a sequence of squares such that the successive squares are neighbours, that is, given any square  $T$  other than  $S$ , there is a sequence of squares

$$S_1, S_2, \dots, S_n \quad \text{with } S_1 = S, S_n = T$$

such that  $S_{i+1}$  is a neighbour of  $S_i$  for all  $1 \leq i < n$ . Then by the above argument, the entries of the squares  $S_1, S_2, \dots, S_n$  are equal to  $m$ . ■

## References

- [Che25] EVAN CHEN. *The OTIS Excerpts*. Available at <https://web.evanchen.cc/excerpts.html>. 2025, pp. vi+289 (cited p. 1)

- [**Eng98**] ARTHUR ENGEL. *Problem-solving strategies*. Problem Books in Mathematics. Springer-Verlag, New York, 1998, pp. x+403. ISBN: 0-387-98219-1 (cited p. [2](#))