Counting in two different ways

MOPSS

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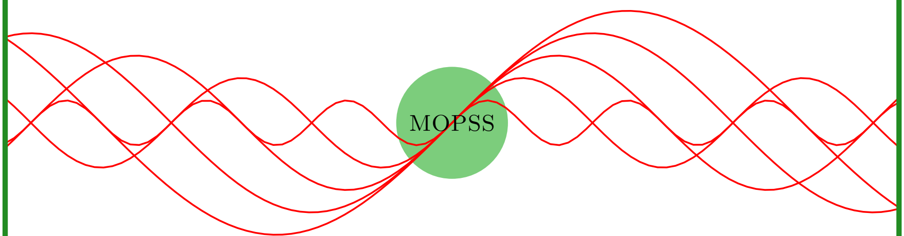
Mathematics Olympiad

Problem Solving Sessions



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https://jpsaha.github.io/MOTP/MOPSS/

Suggested readings

- Evan Chen's advice On reading solutions, available at https://blog.evanchen.cc/2017/03/06/on-reading-solutions/.
- Evan Chen's Advice for writing proofs/Remarks on English, available at https://web.evanchen.cc/handouts/english/english.pdf.
- Notes on proofs by Evan Chen from OTIS Excerpts [**Che25**, Chapter 1].
- Tips for writing up solutions by Edward Barbeau, available at https://www.math.utoronto.ca/barbeau/writingup.pdf.
- Evan Chen discusses why math olympiads are a valuable experience for high schoolers in the post on Lessons from math olympiads, available at https://blog.evanchen.cc/2018/01/05/lessons-from-math-olympiads/.

List of problems and examples

§1 Counting in two different ways

See [**AF13**, §7], [**AE11**, §3.3].

Example 1.1. Find the sum of all distinct four digit numbers that can be formed using the digits $1, 2, 3, 4, 5$, each digit appearing at most once.

Solution 1. Let a, b, c, d be four distinct elements of $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Then among the four digit numbers that can be formed using a, b, c, d with no repeated digits, there are exactly six numbers with i in its j's place for any $i \in \{a, b, c, d\}, j \in \{1, 10, 100, 1000\}$. So the sum of such numbers is equal to $(a + b + c + d) \times (6 + 60 + 600 + 6000) = 6666(a + b + c + d)$. This shows that the required sum is equal to

$$\sum_{\{a,b,c,d\} \subseteq \{1,2,\dots,9\}} 6666(a + b + c + d).$$

Note that

$$\begin{aligned}
\sum_{\substack{A \subseteq \{1,2,\dots,9\} \\ |A|=4}} \sum_{a \in A} a &= \sum_{a=1}^{9} \sum_{\substack{A \subseteq \{1,2,\dots,9\} \\ |A|=4, a \in A}} a \\
&= \sum_{a=1}^{9} \binom{8}{3} a \\
&= 45 \times \binom{8}{3} \\
&= 25200.
\end{aligned}$$

Hence, the required sum is equal to

$$6666 \times 25200.$$

■

Example 1.2 (India BStat-BMath 2014)**.** A class has 100 students. Let a_i, $1 \le i \le 100$, denote the number of friends the i-th student has in the class. For each $0 \le j \le 99$, let c_j denote the number of students having at least j friends. Show that

$$a_1 + a_2 + \cdots + a_{100} = c_1 + c_2 + \cdots + c_{99}.$$

Solution 2. For $1 \le i \le 100$, denote the i-th student by s_i. For $1 \le j \le 99$, let C_j denote the set of students having at least j friends. Note that for any $1 \le i \le 100$,

$$a_i = \sum_{j=1}^{99} 1_{C_j}(s_i)$$

holds, where for $1 \le j \le 99$, 1_{C_j} denotes the map, defined on $\{s_1, s_2, \dots, s_{100}\}$, given by

$$1_{C_j}(s_i) = \begin{cases} 1 & \text{if } s_i \text{ lies in } C_j, \\ 0 & \text{otherwise.} \end{cases}$$

Summing over $1 \le i \le 100$, and interchanging the order of summation, we obtain

$$\begin{aligned} a_1 + a_2 + \cdots + a_{100} &= \sum_{j=1}^{99} \sum_{i=1}^{100} 1_{C_j}(s_i) \\ &= \sum_{j=1}^{99} |\{s_i \mid s_i \in C_j\}| \\ &= \sum_{j=1}^{99} c_j. \end{aligned}$$

This completes the proof. ■

> **Remark 1.** The following is **somewhat naive**, and **does require** additional explanation to be included (at which step(s)?). The following explains (with some effort from readers' end, of course!) why the stated result should hold, and it may also help to arrive at the above solution. However, the following lacks some details.

Walkthrough —

$$
\begin{array}{ccccccccc}
 & c_1 \text{ bullets} & c_2 \text{ bullets} & c_3 \text{ bullets} & \cdots & c_j \text{ bullets} & \cdots & \\
 & \downarrow & \downarrow & \downarrow & \cdots & \downarrow & \cdots & \\
s_1 & \bullet & \bullet & \bullet & \cdots & \bullet & \cdots & \leftarrow a_1 \text{ bullets} \\
s_2 & \bullet & \bullet & \bullet & \cdots & \bullet & \cdots & \leftarrow a_2 \text{ bullets} \\
s_3 & \bullet & \bullet & \bullet & \cdots & \bullet & \cdots & \leftarrow a_3 \text{ bullets} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
s_i & \bullet & \bullet & \bullet & \cdots & \bullet & \cdots & \leftarrow a_i \text{ bullets} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
s_{100} & \bullet & \bullet & \bullet & \cdots & \bullet & \cdots & \leftarrow a_{100} \text{ bullets}
\end{array}
$$

Denote the students by $s_1, s_2, \dots, s_{100}$ and write $s_1, s_2, \dots, s_{100}$ in a vertical column. Then for each $1 \leq i \leq 100$, put a_i bullets next to s_i (as shown above). Then the sum $a_1 + \cdots + a_{100}$ is equal to the total number of bullets. It turns out that the number of bullets in the j-th column is equal to c_j for any $1 \leq j \leq 99$, proving that the total number of bullets is also equal to $c_1 + c_2 + c_3 + \cdots + c_{99}$.

Example 1.3 (India RMO 2019a P6)**.** Suppose 91 distinct positive integers greater than 1 are given such that there are at least 456 pairs among them which are relatively prime. Show that one can find four integers a, b, c, d among them such that $\gcd(a, b) = \gcd(b, c) = \gcd(c, d) = \gcd(d, a) = 1$.

Solution 3. Denote the chosen 91 integers by $n_1, \dots, n_{91}$. For $1 \leq i \leq 91$, let d_i denote the number of integers among them which are relatively prime with n_i. Consider the triples (a, b, c) where a, b, c run over the chosen 91 integers, and satisfy $\gcd(a, b) = \gcd(a, c) = 1$. Counting these triples along their first entry, it follows that the number of such triples is

$$\binom{d_1}{2} + \cdots + \binom{d_{91}}{2}.$$

If the statement to be proved is false, then the given any two distinct integers b, c among them, it can be completed to at most one such triple (a, b, c). This shows that

$$\binom{d_1}{2} + \cdots + \binom{d_{91}}{2} \leq \binom{91}{2},$$

which implies that

$$
\begin{aligned}
\binom{91}{2} &\geq \frac{1}{2}\left(d_1^2 + \cdots + d_{91}^2 - d_1 - \cdots - d_{91}\right) \\
&= \frac{1}{2}\left(d_1^2 + \cdots + d_{91}^2 - 2 \cdot 456\right) \\
&\geq \frac{1}{2}\left(\frac{(d_1 + \cdots + d_{91})^2}{91} - 912\right) \quad \text{(using the Cauchy–Schwartz inequality)}
\end{aligned}
$$

$$= \frac{1}{2}\left(\frac{(2\cdot 456)^2}{91} - 912\right)$$
$$= 456 \cdot 9,$$

and this yields $910 > 912$, which is impossible. This completes the proof. ■

Example 1.4 (India TST 2001?). [Sri14, Chapter 6, Example 9] Let G be a graph on n vertices, having e edges, and no 4-cycles. Show that

$$e \leq \frac{n}{4}(1+\sqrt{4n-3}).$$

Summary — A similar counting as above yields

$$\binom{n}{2} \geq \frac{1}{2}\left(\frac{(2e)^2}{n} - 2e\right),$$

which implies the above bound.

Example 1.5 (IMOSL 1995 N5). At a meeting of $12k$ people, each person exchanges greetings with exactly $3k+6$ others. For any two people, the number who exchange greetings with both is the same. How many people are at the meeting?

Solution 4. Let λ denote the integer such that for any two people, precisely λ persons exchange greetings with them. Counting the number of triples of the form (a, b, c) where a exchanges greetings with b and c, we obtain

$$12k\binom{3k+6}{2} = \lambda\binom{12k}{2},$$

which yields

$$\begin{aligned}\lambda &= \frac{(3k+5)(3k+6)}{12k-1}\\ &= \frac{1}{4}\frac{36k^2+132k+120}{12k-1}\\ &= \frac{1}{4}\left(3k+11+\frac{3k+131}{12k-1}\right)\\ &= \frac{3k+11}{4} + \frac{1}{16}\left(1+\frac{525}{12k-1}\right).\end{aligned}$$

This shows that $12k-1$ divides 525 and the quotient $525/(12k-1)$ is congruent to 3 modulo 4, that is, for some integer $\ell \geq 0$, we have

$$525 = (12k-1)(4\ell+3).$$

Noting that $525 = 3 \cdot 5^2 \cdot 7$, it follows that one of $12k - 1, 4\ell + 3$ is a multiple of 3 and the other one is a multiple of 7. This shows that $12k - 1$ is equal to one of

$$3, 15, 45, 7, 35, 105,$$

and hence $12k - 1 = 36$, which gives $k = 3$. It follows that there are 36 people at the meeting. ■

Example 1.6 (India RMO 2023a P4)**.** The set X of N four-digit numbers formed from the digits $1, 2, 3, 4, 5, 6, 7, 8$ satisfies the following condition: *for any two different digits from* $1, 2, 3, 4, 5, 6, 7, 8$*, there exists a number in* X *which contains both of them.* Determine the smallest possible value of N.

Solution 5. Note that the following set of four-digit numbers satisfy the required condition.

$$\{1234, 5678, 1256, 3478, 1278, 3456\}$$

This shows that the smallest possible value of N is at most 6.

Let X be a set of N four-digit numbers satisfying the given condition. Note that for any integer $1 \leq i \leq 8$, two four-digit numbers containing i as a digit, together contains at most 7 distinct elements of $\{1, 2, \ldots, 8\}$ as digits. Hence, for any $1 \leq i \leq 8$, there are at least three elements in X which contain i as a digit. Counting the number of pairs of the form (i, x), where $1 \leq i \leq 8$ and x is an element of X containing i as a digit, we obtain

$$4N \geq 8 \cdot 3,$$

which yields $N \geq 6$.

This proves that the smallest possible value of N is 6. ■

> **Remark.** Is it helpful to count the pairs of the form $(\{i, j\}, x)$ where $1 \leq i < j \leq 8$ and x is an element of X containing i, j as digits?

References

[**AE11**] Titu Andreescu and Bogdan Enescu. *Mathematical Olympiad treasures*. Second. Birkhäuser/Springer, New York, 2011, pp. viii+253. isbn: 978-0-8176-8252-1; 978-0-8176-8253-8 (cited p. 2)

[**AF13**] T. Andreescu and Z. Feng. *A Path to Combinatorics for Undergraduates: Counting Strategies*. Birkhäuser Boston, 2013. isbn: 9780817681548. url: https://books.google.de/books?id=3mwQBwAAQBAJ (cited p. 2)

[**Che25**] Evan Chen. *The OTIS Excerpts*. Available at https://web.evanchen.cc/excerpts.html. 2025, pp. vi+289 (cited p. 1)

[**Sri14**] Pranav Sriram. "Olympiad Combinatorics". Available at https://www.aops.com/community/c6h601134. 2014 (cited p. 5)