

# Counting via bijections

MOPSS

7 May 2025

## Mathematics Olympiad

Problem Solving Sessions



MOPSS

DEPARTMENT OF MATHEMATICS  
IISER BHOPAL

<https://jpsaha.github.io/MOTP/MOPSS/>

## Suggested readings

- Evan Chen's advice [On reading solutions](https://blog.evanchen.cc/2017/03/06/on-reading-solutions/), available at <https://blog.evanchen.cc/2017/03/06/on-reading-solutions/>.
- Evan Chen's [Advice for writing proofs/Remarks on English](https://web.evanchen.cc/handouts/english/english.pdf), available at <https://web.evanchen.cc/handouts/english/english.pdf>.
- [Notes on proofs](#) by Evan Chen from [OTIS Excerpts](#) [[Che25](#), Chapter 1].
- [Tips for writing up solutions](https://www.math.utoronto.ca/barbeau/writingup.pdf) by Edward Barbeau, available at <https://www.math.utoronto.ca/barbeau/writingup.pdf>.
- Evan Chen discusses why [math olympiads are a valuable experience for high schoolers](#) in the post on [Lessons from math olympiads](#), available at <https://blog.evanchen.cc/2018/01/05/lessons-from-math-olympiads/>.

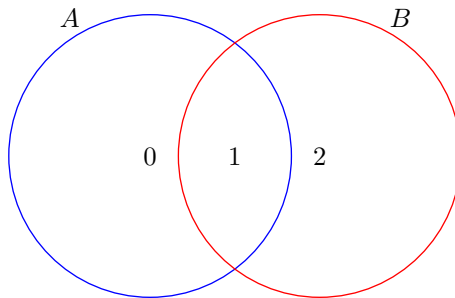


Figure 1: India RMO 1997, Example 1.1

## List of problems and examples

1.1	Example (India RMO 1997 P6)	2
1.2	Example	3
1.3	Example (India RMO 2012a P4, India RMO 2012b P4, India RMO 2012c P4, India RMO 2012d P4)	3
1.4	Example (India RMO 2013b P6)	4
1.5	Example (India RMO 2015c P6)	5
1.6	Example (India RMO 2024b P6)	6

## §1 Counting via bijections

**Example 1.1** (India RMO 1997 P6). Find the number of unordered pairs  $\{A, B\}$  (that is, the pairs  $\{A, B\}$  and  $\{B, A\}$  are considered to be the same) of subsets of an  $n$ -element set  $X$  which satisfy the conditions:

- (a)  $A \neq B$ ,
- (b)  $A \cup B = X$ .

(eg., if  $X = \{a, b, c, d\}$ , then  $\{\{a, b\}, \{b, c, d\}\}$ ,  $\{\{a\}, \{b, c, d\}\}$ ,  $\{\emptyset, \{a, b, c, d\}\}$  are some of the admissible pairs.)

**Walkthrough** — Establish a suitable one-to-one correspondence between the set of the ordered pairs  $(A, B)$  with  $A \cup B = X$  and the set of maps  $f : X \rightarrow \{0, 1, 2\}$ .

**Solution 1.** Note that the ordered pairs  $(A, B)$  with  $A \cup B = X$  are in a one-to-one correspondence with the maps  $f : X \rightarrow \{0, 1, 2\}$ . Such a one-to-one correspondence is given by sending such a pair  $(A, B)$  to  $f : X \rightarrow \{0, 1, 2\}$  taking the values 0, 1, 2 at the subsets  $A \setminus (A \cap B)$ ,  $A \cap B$ ,  $B \setminus (A \cap B)$  of  $X$ , respectively (see Fig. 1). This shows that there are  $3^n$  ordered pairs  $(A, B)$

satisfying  $A \cup B = X$ . Among these pairs, there is only one pair satisfying  $A = B$ , namely, the pair  $(X, X)$ . Hence, the number of unordered pairs  $\{A, B\}$  satisfying  $A \neq B$  and  $A \cup B = X$  is equal to  $\frac{1}{2}(3^n - 1)$ . ■

**Example 1.2.** Let  $S = \{1, 2, \dots, n\}$ . Find the number of unordered pairs  $\{A, B\}$  of subsets of  $S$  such that  $A$  and  $B$  are disjoint, where  $A$  or  $B$  or both may be empty.

**Walkthrough** — Establish a suitable one-to-one correspondence between the set of the ordered pairs  $(A, B)$  of subsets of  $S$ , and the set of maps  $f : X \rightarrow \{0, 1, 2\}$ .

**Solution 2.** Note that the number of ordered pairs  $(A, B)$  of disjoint subsets of  $S$  is equal to the number of functions from  $S$  to  $\{0, 1, 2\}$ , which is equal to  $3^n$ . From such an ordered pair  $(A, B)$ , we get an unordered pair  $\{A, B\}$  as desired. Moreover, an unordered pair  $\{A, B\}$  of disjoint subsets of  $S$ , comes from  $(A, B)$ , and also from  $(B, A)$ . Note that for such an unordered pair  $\{A, B\}$ , the order pairs  $(A, B), (B, A)$  are equal if and only if  $A = B = \emptyset$ . Hence, the number of the unordered pairs satisfying the given conditions is equal to  $1 + \frac{1}{2}(3^n - 1) = \frac{1}{2}(3^n + 1)$ . ■

**Example 1.3** (India RMO 2012a P4, India RMO 2012b P4, India RMO 2012c P4, India RMO 2012d P4).

1. Let  $X = \{1, 2, 3, \dots, 10\}$ . Find the number of pairs  $\{A, B\}$  such that  $A \subseteq X, B \subseteq X, A \neq B$  and  $A \cap B = \{2, 3, 5, 7\}$ .
2. Let  $X = \{1, 2, 3, \dots, 12\}$ . Find the number of pairs  $\{A, B\}$  such that  $A \subseteq X, B \subseteq X, A \neq B$  and  $A \cap B = \{2, 3, 5, 7, 8\}$ .
3. Let  $X = \{1, 2, 3, \dots, 10\}$ . Find the number of pairs of  $\{A, B\}$  such that  $A \subseteq X, B \subseteq X, A \neq B$  and  $A \cap B = \{5, 7, 8\}$ .
4. Let  $X = \{1, 2, 3, \dots, 11\}$ . Find the number of pairs of  $\{A, B\}$  such that  $A \subseteq X, B \subseteq X, A \neq B$  and  $A \cap B = \{4, 5, 7, 8, 9, 10\}$ .

**Walkthrough** — Establish a suitable one-to-one correspondence between the set of ordered pairs  $(A, B)$  of subsets of  $\{1, 2, \dots, 10\}$  satisfying  $A \cap B = \{2, 3, 5, 7\}$ , and the set of maps  $f : \{1, 2, \dots, 10\} \setminus \{2, 3, 5, 7\} \rightarrow \{0, 1, 2\}$ .

**Solution 3.** Note that given a subset  $Y$  of a set  $X$ , the ordered pairs  $(A, B)$  of subsets of  $X$  with  $A \cap B = Y$  are in one-to-one correspondence with the maps from  $X \setminus Y$  to  $\{0, 1, 2\}$  (where the inverse images of 0, 1, 2 correspond to  $A \setminus (A \cap B), B \setminus (A \cap B), X \setminus (A \cup B)$ , respectively). Moreover, if  $(A, B)$  is such

an order pair, then  $(A, B) = (B, A)$  holds if and only if  $A = B = Y$ . It follows that the number of ordered pairs  $\{A, B\}$  satisfying  $A \cap B = Y$  and  $A \neq B$  is equal to  $\frac{1}{2}(3^{|X \setminus Y|} - 1)$  if  $X \setminus Y$  is finite.

The number of pairs of the given type are  $3^6 - 1, 3^7 - 1, 3^7 - 1, 3^5 - 1$ . ■

**Example 1.4 (India RMO 2013b P6).** For a natural number  $n$ , let  $T(n)$  denote the number of ways we can place  $n$  objects of weights  $1, 2, \dots, n$  on a balance such that the sum of the weights in each pan is the same. Prove that  $T(100) > T(99)$ .

### Walkthrough —

- Try to find out a placing of the weights  $1, 2, \dots, 99$  on a balance such that the weights on the pans are the same.
- How about putting  $1, 2, 3, \dots, 49$  on one pan, and  $51, 52, \dots, 99$  on the other? Note that the weights on the pans are the same. However, the object of weight 50 has not been included. This can be resolved by placing the weights  $(\{1, 2, \dots, 49\} \setminus \{25\}) \cup \{50\}$  on one pan, the weights  $\{25\} \cup \{51, 52, \dots, 99\}$  on the other.
- Does the above help in finding an injective map from the set of all possible placements of the weights  $1, 2, \dots, 99$  on a balance satisfying the required condition, to the set of all possible placements of the weights  $1, 2, \dots, 100$  on a balance satisfying the required condition? Can one also have in addition that this map is not surjective?

**Solution 4.** Let  $S_n$  denote the set of pairs  $(A, B)$  where  $A, B$  are disjoint subsets of  $\{1, 2, \dots, n\}$  such that their union is  $\{1, 2, \dots, n\}$  and the sum of elements of  $A$  is equal to the sum of elements of  $B$ . To establish  $T(99) < T(100)$ , it suffices to construct an injective map  $f : S_{99} \rightarrow S_{100}$  which is not surjective. Define  $f$  by

$$f(A, B) = \begin{cases} (A \cup \{100\} \setminus \{50\}, B \cup \{50\}) & \text{if } 50 \in A, \\ (A \cup \{50\}, B \cup \{100\} \setminus \{50\}) & \text{if } 50 \in B. \end{cases}$$

Note that  $f$  is well-defined (since given an element  $(A, B)$  of  $S_{99}$ , either  $A$  contains 50 or  $B$  contains 50).

**Claim —** The map  $f$  is injective.

*Proof of the Claim.* Let  $(A, B), (C, D)$  be two elements of  $S_{99}$  having the same image under  $f$ .

Let us consider the case that  $A$  contains 50. Since the second coordinate of  $f(C, D) = f(A, B)$  contains 50, it follows that  $C$  contains 50. Considering the second coordinate of  $f(A, B)$  and that of  $f(C, D)$ , we obtain  $B \cup \{50\} = D \cup \{50\}$ , which yields  $B = D$  (since  $50 \notin B$  and  $50 \notin D$ ). This also gives  $A = C$ .

If  $B$  contains 50, then a similar argument shows that  $A = C$ , and this gives that  $B = D$ .  $\square$

**Claim** — The map  $f : S_{99} \rightarrow S_{100}$  is not surjective.

*Proof of the Claim.* To show that  $f$  is not surjective, it suffices to find a pair  $(A, B)$  of subsets of  $\{1, 2, \dots, 100\}$  such that  $A \cap B = \emptyset$ ,  $A \cup B = \{1, 2, \dots, 100\}$ , and  $A$  contains 50 and 100.

Note that for the sum of the elements of any of the following 50 sets

$$\{1, 100\}, \{2, 99\}, \{3, 98\}, \dots, \{50, 51\}$$

is equal to 101. So the pair  $(A, B)$  belongs to  $S_{100}$ , where

$$\begin{aligned} A &= \{1, 100\} \cup \{2, 99\} \cup \dots \cup \{24, 77\} \cup \{50, 51\}, \\ B &= \{26, 75\} \cup \{27, 74\} \cup \dots \cup \{49, 52\} \cup \{25, 76\}. \end{aligned}$$

Moreover, the pair  $(A, B)$  does not belong to  $f(S_{99})$ .  $\square$

Using the above Claims, it follows that  $T(99) < T(100)$ .  $\blacksquare$

**Example 1.5 (India RMO 2015c P6).** Let  $S = \{1, 2, \dots, n\}$  and let  $T$  be the set of all ordered triples of subsets of  $S$ , say  $(A_1, A_2, A_3)$ , such that  $A_1 \cup A_2 \cup A_3 = S$ . Determine in terms of  $n$ ,

$$\sum_{(A_1, A_2, A_3) \in T} |A_1 \cap A_2 \cap A_3|,$$

where  $|X|$  denotes the number of elements in the set  $X$ .

**Walkthrough** — Find a suitable one-to-one correspondence between the triples of the subsets of  $S$ , whose union is  $S$  and intersection is a given subset  $X$  of  $S$ , and the maps from  $S \setminus X$  to  $\{1, 2, \dots, 6\}$ .

**Solution 5.** Note that for  $0 \leq r \leq n$ , the set  $S$  has  $\binom{n}{r}$  many subsets with  $r$  elements. Also note that for  $0 \leq r < n$ , and given a subset  $X$  of  $S$  with  $r$  elements, the number of ordered tuples  $(A_1, A_2, A_3)$  of subsets of  $S$  with  $A_1 \cap A_2 \cap A_3 = X$  and  $A_1 \cup A_2 \cup A_3 = S$  is equal to the number of maps from  $S \setminus X$  to  $\{1, 2, 3, 4, 5, 6\}$ , which is equal to  $6^{n-r}$ . Moreover, there is only one ordered tuples  $(A_1, A_2, A_3)$  of subsets of  $S$  with  $A_1 \cap A_2 \cap A_3 = S$ . So

$$\sum_{(A_1, A_2, A_3) \in T} |A_1 \cap A_2 \cap A_3| = \sum_{r=0}^{n-1} \binom{n}{r} 6^{n-r} r + n$$

$$\begin{aligned}
&= \sum_{r=1}^n \binom{n}{r} 6^{n-r} r \\
&= n \sum_{r=0}^{n-1} \binom{n-1}{r} 6^{n-1-r} \\
&= n 7^{n-1},
\end{aligned}$$

where the final equality is obtained by applying the binomial theorem, and the second last equality is obtained by counting the size of the following set in two different ways.

$$\{(A, a, f) \mid A \subseteq S, a \in A, f \text{ is a map from } S \setminus A \text{ to } \{1, 2, 3, 4, 5, 6\}\}$$

■

**Example 1.6 (India RMO 2024b P6).** Let  $n \geq 2$  be a positive integer. Call a sequence  $a_1, a_2, \dots, a_k$  of integers an  $n$ -chain if  $1 = a_1 < a_2 < \dots < a_k = n$ ,  $a_i$  divides  $a_{i+1}$  for all  $i$ ,  $1 \leq i \leq k-1$ . Let  $f(n)$  be the number of  $n$ -chains where  $n \geq 2$ . For example,  $f(4) = 2$  corresponds to the 4-chains  $\{1, 4\}$  and  $\{1, 2, 4\}$ . Prove that  $f(2^m \cdot 3) = 2^{m-1}(m+2)$  for every positive integer  $m$ .

### Walkthrough —

(a) Let us determine  $f(2), f(4), f(8), \dots$

- Note that  $f(2) = 1$  since  $\{1, 2\}$  is the only 2-chain.
- Note that  $f(2^2) = 2$  since  $\{1, 2^2\}, \{1, 2, 2^2\}$  are the only  $2^2$ -chains.
- Note that  $f(2^3) = 2^2$  since  $\{1, 2^3\}, \{1, 2, 2^3\}, \{1, 2^2, 2^3\}, \{1, 2, 2^2, 2^3\}$  are the only  $2^3$ -chains.

The above examples suggest that  $f(2^m) = 2^{m-1}$  for any integer  $m \geq 1$ . However, this does require a proof, which we will do in a while.

(b) Let us determine  $f(2 \cdot 3), f(2^2 \cdot 3), f(2^3 \cdot 3), \dots$

- Note that  $f(2 \cdot 3) = 3$  since there are only 3 many  $2 \cdot 3$ -chains.  
Here is the  $2 \cdot 3$ -chain containing **3** as the smallest multiple of 3.  
–  $\{1, \mathbf{3}, 2 \cdot 3\}$ ,  
Here are the  $2 \cdot 3$ -chains containing **2 · 3** as the smallest multiple of 3.  
–  $\{1, 2, \mathbf{2 \cdot 3}\}$ ,  
–  $\{1, \mathbf{2 \cdot 3}\}$
- Note that  $f(2^2 \cdot 3) = 2^1 \cdot (2+2)$  since there are  $2^1 \cdot (2+2)$  many  $2^2 \cdot 3$ -chains.  
Here are the  $2^2 \cdot 3$ -chains containing **3** as the smallest multiple of 3.  
–  $\{1, \mathbf{3}, 2^2 \cdot 3\}$ ,

$$- \{1, \mathbf{3}, 2 \cdot 3, 2^2 \cdot 3\},$$

Here are the  $2^2 \cdot 3$ -chains containing  $\mathbf{2 \cdot 3}$  as the smallest multiple of 3.

$$- \{1, 2, \mathbf{2 \cdot 3}, 2^2 \cdot 3\},$$

$$- \{1, \mathbf{2 \cdot 3}, 2^2 \cdot 3\},$$

Here are the  $2^2 \cdot 3$ -chains containing  $\mathbf{2^2 \cdot 3}$  as the smallest multiple of 3.

$$- \{1, 2^2, \mathbf{2^2 \cdot 3}\},$$

$$- \{1, 2, 2^2, \mathbf{2^2 \cdot 3}\},$$

$$- \{1, \mathbf{2^2 \cdot 3}\},$$

$$- \{1, 2, \mathbf{2^2 \cdot 3}\},$$

- Note that  $f(2^3 \cdot 3) = 2^2 \cdot (3 + 2)$  since there are  $2^2 \cdot (3 + 2)$  many  $2^3 \cdot 3$ -chains.

Here are the  $2^3 \cdot 3$ -chains containing  $\mathbf{3}$  as the smallest multiple of 3.

$$- \{1, \mathbf{3}, 2^3 \cdot 3\},$$

$$- \{1, \mathbf{3}, 2 \cdot 3, 2^3 \cdot 3\},$$

$$- \{1, \mathbf{3}, 2^2 \cdot 3, 2^3 \cdot 3\},$$

$$- \{1, \mathbf{3}, 2 \cdot 3, 2^2 \cdot 3, 2^3 \cdot 3\},$$

Here are the  $2^3 \cdot 3$ -chains containing  $\mathbf{2 \cdot 3}$  as the smallest multiple of 3.

$$- \{1, 2, \mathbf{2 \cdot 3}, 2^3 \cdot 3\},$$

$$- \{1, 2, \mathbf{2 \cdot 3}, 2^2 \cdot 3, 2^3 \cdot 3\},$$

$$- \{1, \mathbf{2 \cdot 3}, 2^3 \cdot 3\},$$

$$- \{1, \mathbf{2 \cdot 3}, 2^2 \cdot 3, 2^3 \cdot 3\},$$

Here are the  $2^3 \cdot 3$ -chains containing  $\mathbf{2^2 \cdot 3}$  as the smallest multiple of 3.

$$- \{1, 2^2, \mathbf{2^2 \cdot 3}, 2^3 \cdot 3\},$$

$$- \{1, 2, 2^2, \mathbf{2^2 \cdot 3}, 2^3 \cdot 3\},$$

$$- \{1, \mathbf{2^2 \cdot 3}, 2^3 \cdot 3\},$$

$$- \{1, 2, \mathbf{2^2 \cdot 3}, 2^3 \cdot 3\},$$

Here are the  $2^3 \cdot 3$ -chains containing  $\mathbf{2^3 \cdot 3}$  as the smallest multiple of 3.

$$- \{1, 2^3, \mathbf{2^3 \cdot 3}\},$$

$$- \{1, 2, 2^3, \mathbf{2^3 \cdot 3}\},$$

$$- \{1, 2^2, 2^3, \mathbf{2^3 \cdot 3}\},$$

$$- \{1, 2, 2^2, 2^3, \mathbf{2^3 \cdot 3}\},$$

$$- \{1, \mathbf{2^3 \cdot 3}\},$$

$$- \{1, 2, \mathbf{2^3 \cdot 3}\},$$

$$- \{1, 2^2, \mathbf{2^3 \cdot 3}\},$$

$$- \{1, 2, 2^2, 2^3 \cdot 3\},$$

The above examples suggest that  $f(2^m \cdot 3) = 2^{m-1}(m+2)$  for any integer  $m \geq 1$ . Certainly, this does require a proof, which we will do in a while.

### Solution 6.

**Claim** — For any integer  $m \geq 1$ , there are precisely  $2^{m-1}$  many  $2^m$ -chains. In other words,  $f(2^m) = 2^{m-1}$  for any integer  $m \geq 1$ .

*Proof of the Claim.* In a  $2^m$ -chain, the smallest term is 1 and the largest term is  $2^m$ . The remaining terms are some (or none if the chain has only two terms) of the powers of 2 lying between 1 and  $2^m$ . Thus, to form a  $2^m$ -chain, one needs to determine the possibilities for the terms other than the smallest and the largest one. To do so, from the remaining  $m-1$  many powers of 2 lying between 1 and  $2^m$ , we can choose as many and arrange them in an increasing order. By the multiplication principle, this can be done in  $2^{m-1}$  ways. This proves the Claim.  $\square$

**Claim** — Let  $m$  be a positive integer, and  $0 \leq k < m$  be an integer. The number of  $2^m \cdot 3$ -chains, which contain  $2^k \cdot 3$  as the smallest multiple of 3, is  $2^{m-1}$ .

*Proof of the Claim.* If  $k = 0$ , then the  $2^m \cdot 3$ -chains are in one-to-one correspondence with the  $2^m$ -chains, and hence, it follows that  $f(2^m \cdot 3) = f(2^m) = 2^{m-1}$ .

Let us assume that  $1 \leq k < m$ . Then it is clear that

$$f(2^m \cdot 3) = f(2^k)f(2^{m-k}) + f(2^k)f(2^{m-k}) = 2f(2^k)f(2^{m-k}).$$

Using the first Claim, it follows that

$$f(2^m \cdot 3) = 2^{m-1}.$$

This completes the proof.  $\square$

**Claim** — Let  $m$  be a positive integer. The number of  $2^m \cdot 3$ -chains, which contain  $2^m \cdot 3$  as the smallest multiple of 3, is  $2^m$ .

*Proof.* Note that the  $2^m \cdot 3$ -chains, which contain  $2^m \cdot 3$  as the smallest multiple of 3 and also contain  $2^m$ , are in one-to-one correspondence with the  $2^m$ -chains. Moreover, the  $2^m \cdot 3$ -chains, which contain  $2^m \cdot 3$  as the smallest multiple of 3 and does not contain  $2^m$ , are also in one-to-one correspondence with the  $2^m$ -chains. It follows that the number of  $2^m \cdot 3$ -chains, which contain  $2^m \cdot 3$  as the smallest multiple of 3, is  $2 \cdot f(2^m) = 2^m$ .  $\square$



Combining the two Claims above, it follows that

$$f(2^m \cdot 3) = \sum_{0 \leq k < m} 2^{m-1} + 2^m = 2^{m-1}(m+2).$$

This completes the proof. ■

## References

- [Che25] EVAN CHEN. *The OTIS Excerpts*. Available at <https://web.evanchen.cc/excerpts.html>. 2025, pp. vi+289 (cited p. 1)