Counting

MOPSS

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Mathematics Olympiad

Problem Solving Sessions

MOPSS

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https://jpsaha.github.io/MOTP/MOPSS/

Suggested readings

- Evan Chen's advice On reading solutions, available at https://blog.evanchen.cc/2017/03/06/on-reading-solutions/.
- Evan Chen's Advice for writing proofs/Remarks on English, available at https://web.evanchen.cc/handouts/english/english.pdf.
- Notes on proofs by Evan Chen from OTIS Excerpts [**Che25**, Chapter 1].
- Tips for writing up solutions by Edward Barbeau, available at https://www.math.utoronto.ca/barbeau/writingup.pdf.
- Evan Chen discusses why math olympiads are a valuable experience for high schoolers in the post on Lessons from math olympiads, available at https://blog.evanchen.cc/2018/01/05/lessons-from-math-olympiads/.

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§1 Counting

Example 1.1. Show that an n-gon has $n(n-3)/2$ diagonals.

Solution 1. Let $A_1 \cdots A_n$ be an n-gon. To get a diagonal passing through A_1, we can join it to any vertex of the n-gon other than itself and its two adjacent vertices, i.e., we can join A_1 to $n-1-2=n-3$ vertices of the n-gon to get all the diagonals passing through A_1. A similar statement holds for any vertex of the n-gon. In this way, we would get $n(n-3)$ diagonals. However, note that we have counted each diagonal twice, for example, the diagonal A_iA_j is counted once while counting diagonals passing through A_i and one more time while counting diagonals passing through A_j. So an n-gon has $n(n-3)/2$ diagonals. ■

Example 1.2. In how many ways can we fill a bag with 50 fruits using bananas and apples such that number of bananas is even, number of apples is a multiple of 3?

Solution 2. We need to determine in how many ways a nonnegative multiple of 2 and a nonnegative multiple of 3 add up to 50, i.e., we need to solve the equation $2x + 3y = 50$ in the nonnegative integers. Note that this equation is equivalent to $2x + 3y = 50 \cdot 3 - 50 \cdot 2$, that is, $2(x + 50) = 3(50 - y)$. Since $2, 3$ are coprime, it follows that 3 divides $x + 50$ and 2 divides $50 - y$. So the solutions to $2x + 3y = 50$ in the integers are of the form $(3k - 50, 50 - 2k)$ for some integer k. Note that both of $3k - 50, 50 - 2k$ are nonnegative if and only if $17 \leq k \leq 25$, that is, $3k - 50$ and $50 - 2k$ are nonnegative for 9 values of k. Consequently, under the given conditions, the bag can be filled in nine ways. ■

Example 1.3. [PK74, Problem 46.1] In a tennis tournament, there are $2n$ participants. In the first round of the tournament each participant plays just once, so there are n games, each occupying a pair of players. Show that the pairing for the first round can be arranged in exactly

$$1 \times 3 \times 5 \times 7 \times 9 \times \cdots \times (2n - 1)$$

different ways.

Solution 3. The pairing can be arranged in

$$\begin{aligned}
&\frac{1}{n!} \times \binom{2n}{2} \cdot \binom{2n-2}{2} \cdot \cdots \cdot \binom{4}{2} \cdot \binom{2}{2} \\
&= \frac{1}{n!} \times \frac{(2n)(2n-1)}{2} \times \frac{(2n-2)(2n-3)}{2} \times \cdots \times \frac{4 \cdot 3}{2} \times \frac{2 \cdot 1}{2} \\
&= \frac{1}{n!} \times n! \times (2n-1) \times (2n-3) \times \cdots \times 3 \times 1 \\
&= (2n-1) \times (2n-3) \times \cdots \times 3 \times 1
\end{aligned}$$

ways. ■

Example 1.4 (India RMO 1993 P3). Suppose $A_1 A_2 \cdots A_{20}$ is a 20-sided regular polygon. How many non-isosceles (scalene) triangles can be formed whose vertices are among the vertices of the polygon but whose sides are not the sides of the polygon?

Walkthrough —

(a) Determine the number of the triangles with vertices among those of $A_1 A_2 \cdots A_{20}$.

(b) Determine the number of the triangles with vertices among those of $A_1 A_2 \cdots A_{20}$, which share at most two sides with the polygon.

> **(c)** Determine the number of the triangles with vertices among those of $A_1A_2\cdots A_{20}$, which share no side with the polygon.
>
> **(d)** Observe that there are no equilateral triangles among them (why?), and determine the number of isosceles triangles among them.

Solution 4. There are $\binom{20}{3}$ triangles whose vertices are among the vertices of the polygon. Among them, there are 20 triangles that share two sides with the polygon and there are $20 \times (20-4)$ triangles that share only side with the polygon. So there are $\binom{20}{3} - 20 - 20 \times (20-4)$ triangles whose vertices are among the vertices of the polygon, but whose sides are not the sides of the polygon. Among these, there are no equilateral triangles and there are $20 \times \left(\frac{1}{2}(20-2) - 1\right) = 160$ triangles that are isosceles. So there are

$$\binom{20}{3} - 20 - 20 \times (20-4) - 160 = 1140 - 500 = 640$$

scalene triangles whose vertices are among the vertices of the polygon but whose sides are not the sides of the polygon. ■

Example 1.5 (India RMO 1995 P6)**.** Suppose $A_1A_2\cdots A_{21}$ is a 21-sided regular polygon inscribed in a circle with centre O. How many triangles $A_iA_jA_k$, $1 \leq i < j < k \leq 21$, contain the centre point O in their interior?

Solution 5. We first count the number of triangles with A_1 as one of the vertices. Suppose $A_1A_iA_j$ is a triangle with O in its interior. Note that both of i, j cannot be > 11. Renaming the vertices if necessary, assume that $2 \leq i \leq 11$. Let Γ denote the circumcircle of the polygon, and let A' (resp. A_i') denote the diametrically opposite point of A (resp. A_i'). Then A_j has to lie in the small arc $A'A_i'$. Note that there are $(i-1)$ vertices of the 21-sided regular polygon that lie within the small arc $A'A_i'$. So there are

$$(2-1) + (3-1) + \cdots + (11-1) = 1 + 2 + \cdots + 10 = 55$$

triangles containing O in their interior and passing through A_1. Hence the total number of triangles satisfying the required condition is equal to $\frac{1}{3} \times 55 \times 21 = 385$. ■

Example 1.6 (India RMO 2000 P4)**.** All the 7-digit numbers containing each of the digits $1, 2, 3, 4, 5, 6, 7$ exactly once, and not divisible by 5, are arranged in increasing order. Find the 2000-th number in this list.

Solution 6. In this list, first $6! - 5! = 600$ numbers has 1 as their first digit. Similarly, the next 600 numbers begin with 2 and next to them, there are exactly 600 numbers that begin with 3. The next $5! - 4! = 96$ numbers begin with 41, and the next $5! - 4! = 96$ numbers begin with 42. Note that

$2000 - 600 - 600 - 600 - 96 - 96 = 8$, and hence we need to find the 8-th among the numbers in this list that begin with 431. Among them, there are $3! - 2! = 4$ numbers that begin with $4, 3, 1, 2$. The next few numbers are $4315267, 4315276, 4315627, 4315672, 4315726, 4315762$. Hence, 2000th number in this list is 4315672. ■

Example 1.7 (India RMO 2001 P4)**.** Consider an $n \times n$ array of numbers

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{pmatrix}.$$

Suppose each row consists of the n numbers $1, 2, \dots, n$ in some order and $a_{ij} = a_{ji}$ for $i = 1, 2, \dots, n$ and $j = 1, 2, \dots, n$. If n is odd, prove that the numbers $a_{11}, a_{22}, a_{33}, \dots, a_{nn}$ are $1, 2, \dots, n$ in some order.

Solution 7. Let $1 \leq k \leq n$ be an integer. Since a_{ij} is equal to a_{ji} for all $1 \leq i, j \leq n$, the number of occurences of k outside the diagonal is even. Since k appears in each row exactly once, there are total n many k's in this array. So the number of occurences of k on the diagonal differs from n by an even number. Since n is odd, it follows that k appears at least once on the diagonal, which holds for all $1 \leq k \leq n$. Hence, the numbers $a_{11}, a_{22}, a_{33}, \dots, a_{nn}$ are equal to $1, 2, \dots, n$ in some order. ■

Example 1.8 (India BMath 2005)**.** In how many ways an $n \times n$ chessboard can be filled with ± 1 so that the product of the entries in each row and each column equals -1?

Solution 8. Let us establish the following Claim.

> **Claim** — Any filling of the upper left $(n-1) \times (n-1)$-square (denoted by S) by ± 1 can be completed uniquely to a filling of the full chessboard by ± 1 so that the required condition holds.

Proof of the Claim. Note that any filling of the upper left $(n-1) \times (n-1)$-square by ± 1 can be uniquely completed to a filling of the $n \times n$ chessboard except the lower right square such that the product of the entries in each row and each column, except for the last row and the last column, is equal to -1. Thus it remains to show that given such a filling of an $n \times n$ chessboard except its lower right square, there is a unique way to fill the lower right square such that the product of the entries in each row and each column is equal to -1, which would follow if we prove that the product (say p_1) of the entries of the first $(n-1)$ squares of the bottom row is equal to the product (say p_2) of the entries of the first $(n-1)$ squares of the right column. Let P denote the product of all the entries of the upper left $(n-1)\times(n-1)$ square. Note that $p_1 \times P$ is equal to $(-1)^{n-1}$ and $p_2 \times P$ is equal to $(-1)^{n-1}$. So p_1, p_2 are equal as P is nonzero. This proves the claim. □

By the above Claim, an $n\times n$ chessboard can be filled in $2^{(n-1)^2}$ ways so that required condition holds. ■

Example 1.9 (India RMO 2005 P4)**.** Find the number of all 5-digit numbers (in base 10) each of which contains the block 15 and is divisible by 15. (For example, $31545, 34155$ are two such numbers.)

Solution 9. Note that any such number ends with 5 or 0, so is one of the following types: $abc15, ab150, ab155, a15b0, a15b5, 15ab0, 15ab5$. The 5-digit numbers of the form $abc15$ which are divisible by 5 are $10215, 10515, 10815, \dots, 99915$, so there are $1+\frac{1}{3}(999-102) = 300$ such numbers. The 5-digit numbers of the form $ab150$, divisible by 3 are $12150, 15150, 18150, \dots, 99150$, so there are $1+\frac{1}{3}(99-12) = 30$ such numbers. Similarly, there are $1+\frac{1}{3}(97-10) = 30$ five digit numbers of the form $ab150$, divisible by 3. Also there are 30 numbers of the each of the following forms: $a15b0, a15b5$. For the form $15ab0$ (resp. $15ab5$), we get $1+\frac{1}{3}(99-00) = 34$ (resp. $1+\frac{1}{3}(97-01) = 33$) numbers. So there are exactly

$$300 + 4\cdot 30 + 34 + 33 = 487$$

many 5-digit numbers which contain the block 15 and are divisible by 15. ■

Example 1.10 (India RMO 2007 P4)**.** How many 6-digit numbers are there such that

(a) the digits of each number are all from the set $\{1,2,3,4,5\}$,

(b) any digit that appears in the number appears at least twice?

For example, 225252 is valid while 222133 is not.

Solution 10. Such a 6-digit number is formed

1. using a,a,a,a,a,a for some $a\in\{1,2,3,4,5\}$, or
2. using a,a,a,a,b,b or a,a,b,b,b,b for some $a,b\in\{1,2,3,4,5\}$ with $a\neq b$, or
3. using a,a,a,b,b,b for some $a,b\in\{1,2,3,4,5\}$ with $a\neq b$, or
4. using a,a,b,b,c,c for three distinct elements $a,b,c\in\{1,2,3,4,5\}$.

So there are

$$\begin{aligned} &5+\binom{5}{2}\left(\frac{6!}{2!4!}+\frac{6!}{2!4!}\right)+\binom{5}{2}\frac{6!}{3!3!}+\binom{5}{3}\frac{6!}{2!2!2!} \\ &=5+\binom{5}{2}(15+15+20+90) \\ &=1405 \end{aligned}$$

many 6-digit numbers satisfying the given conditions. ■

Example 1.11 (India RMO 2008 P4)**.** Find the number of all 6-digit natural numbers such that the sum of their digits is 10 and each of the digits $0,1,2,3$ occurs at least once in them.

Solution 11. If $0,1,2,3,a,b$ denotes the digits of a 6-digit number such that the sum of their digits is 10, then $a+b=4$, *i.e.*, $\{a,b\}$ is equal to $\{0,4\},\{1,3\}$ or $\{2,2\}$. So the digits of the 6-digit numbers satisfying the given conditions are $0,0,1,2,3,4$, or $0,1,1,2,3,3$, or $0,1,2,2,2,3$. So the number of the 6-digit numbers satisfying the given conditions is

$$\left(\frac{6!}{2!}-5!\right)+\left(\frac{6!}{2!2!}-\frac{5!}{2!2!}\right)+\left(\frac{6!}{3!}-\frac{5!}{3!}\right)=240+150+100=490.$$

■

Example 1.12 (India RMO 2010 P3)**.** Find the number of 4-digit numbers (in base 10) having non-zero digits and which are divisible by 4 but not by 8.

Solution 12. For any $1\le a,b,c\le 9$, the four consecutive even integers

$$1000a+100b+10c+2, 1000a+100b+10c+4,$$

$$1000a+100b+10c+6, 1000a+100b+10c+8$$

are congruent to $0,2,4,6 \pmod 8$ in some order. So exactly one of them (which is congruent to 4 modulo 8) is divisible by 4 but not by 8. Hence for any three integers $1\le a,b,c\le 9$, there is a unique nonzero digit d (that is, $1\le d\le 9$) such that $1000a+100b+10c+d$ is divisible by 4, but not by 8. So there are $9\cdot 9\cdot 9=729$ four-digit numbers (in base 10) with nonzero digits which are divisible by 4, but not by 8. ■

Example 1.13 (India RMO 2011b P4)**.** Find the number of 4-digit numbers with distinct digits chosen from the set $\{0,1,2,3,4,5\}$ in which no two adjacent digits are even.

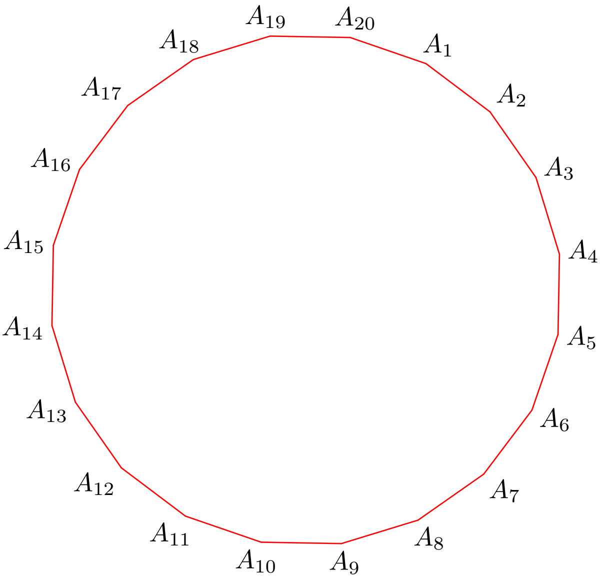

Figure 1: India RMO 2011, Example 1.14

Solution 13. The 4-digit numbers whose no two adjacent digits are even are of the form

$$eooo, eooe, eoeo, oeoo, oeoe, ooeo, oooe$$

where e (resp. o) denotes the evenness (resp. oddness) of the corresponding digit. The numbers of 4-digit numbers of these types with distinct digits chosen from $\{0, 1, 2, 3, 4, 5\}$ and not containing two even numbers as adjacent digits are

$$2 \cdot 3!, 2 \cdot 3 \cdot 2 \cdot 2, 2 \cdot 3 \cdot 2 \cdot 2, 3 \cdot 3!, 3 \cdot 2 \cdot 3 \cdot 2, 3 \cdot 3!, 3 \cdot 3!$$

respectively. Hence, there are

$$2 \cdot 3! + 2 \cdot 3 \cdot 2 \cdot 2 + 2 \cdot 3 \cdot 2 \cdot 2 + 3 \cdot 3! + 3 \cdot 2 \cdot 3 \cdot 2 + 3 \cdot 3! + 3 \cdot 3! = 150$$

4-digit numbers satisfying the given conditions. ■

Example 1.14 (India RMO 2011a P4)**.** Consider a 20-sided convex polygon K, with vertices $A_1, A_2, \ldots, A_{20}$ in that order. Find the number of ways in which three sides of K can be chosen so that every pair among them has at least two sides of K between them. (For example $(A_1A_2, A_4A_5, A_{11}A_{12})$ is an admissible triple while $(A_1A_2, A_4A_5, A_{19}A_{20})$ is not.)

Solution 14. In the following, by an admissible triple, we mean a three-element subset of the set of sides of K, satisfying the required condition. Thus, a triple is **not** ordered. Note that the pair (A_1A_2, A_4A_5) can be completed to the following admissible triples

$$(A_1A_2, A_4A_5, A_7A_8), (A_1A_2, A_4A_5, A_8A_9), \dots, (A_1A_2, A_4A_5, A_{18}A_{19}),$$

and these are all the admissible triples containing the pair (A_1A_2, A_4A_5). Thus there are exactly $18 - 7 + 1 = 12$ admissible triples containing the pair (A_1A_2, A_4A_5). More generally, for any $4 \le i \le 15$, there are exactly $18 - (i + 3) + 1 = 16 - i$ admissible triples containing the pair (A_1A_2, A_iA_{i+1}). Thus it follows that there are exactly

$$\sum_{i=4}^{15} (16 - i) = 1 + 2 + \dots + 12 = 78$$

admissible triples containing the side A_1A_2. So the required number of admissible triples is $\frac{78 \cdot 20}{3} = 520$. ∎

Example 1.15 (India RMO 2013e P3)**.** A finite non-empty set of integers is called 3-*good* if the sum of its elements is divisible by 3. Find the number of 3-good subsets of $\{0, 1, 2, \dots, 9\}$.

Solution 15. For any subset of S of $\{0, 1, 2, \dots, 9\}$, denote by s_0 (resp. s_1, s_2) the number of elements S which are congruent to 0 (resp. $1, 2$) modulo 3. Note that a nonempty subset S of $\{0, 1, 2, \dots, 9\}$ is 3-good if and only if $0 \cdot s_0 + 1 \cdot s_1 + 2 \cdot s_2 \equiv 0 \bmod 3$, that is, $s_1 \equiv s_2 \bmod 3$. The elements of $\{0, 1, 2, \dots, 9\}$ congruent to 0 (resp. 1, 2) modulo 3 are $0, 3, 6, 9$ (resp. $1, 4, 7$, and $2, 5, 8$). So the number of 3-good subsets of $\{0, 1, 2, \dots, 9\}$ is equal to

$$\begin{aligned} &\binom{3}{0}^2 (2^4 - 1) + \binom{3}{1}^2 2^4 + \binom{3}{2}^2 2^4 + \binom{3}{3}^2 2^4 + \binom{3}{0}\binom{3}{3} 2^4 + \binom{3}{3}\binom{3}{0} 2^4 \\ &= 2^4(1 + 9 + 9 + 1 + 1 + 1) - 1 \\ &= 351. \end{aligned}$$

∎

Example 1.16. Let n, m be positive integers. Show that the number of solutions of

$$x_1 + x_2 + \dots + x_m = n \tag{1}$$

in positive integers (resp. nonnegative integers) is equal to $\binom{n-1}{m-1}$ (resp. $\binom{n+m-1}{m-1} = \binom{n+m-1}{n}$).

Solution 16. Note that the solutions of $x_1 + \dots + x_m = n$ in nonnegative integers are in one-to-one correspondence with the solutions of $y_1 + \dots + y_m = n + m$ in positive integers. Indeed, such a correspondence is obtained by sending a solution $(a_1, \ldots, a_m)$ of the first equation in nonnegative integers to $(a_1 + 1, \ldots, a_m + 1)$, which is a solution of the second equation in positive integers. Thus it suffices to show that the number of solutions of Eq. (1) in positive integers is as stated. We represent a positive integer k as a string of k strokes, for example, 5 as $|||||$. Note that a solution of Eq. (1) in positive integers corresponds to placing $m-1$ plus signs among the spaces between a string of n strokes. Since there are $n-1$ spaces among n strokes, the number of solutions of Eq. (1) in positive integers is equal to $\binom{n-1}{m-1}$. ■

Example 1.17 (India RMO 2013c P1)**.** Find the number of eight-digit numbers the sum of whose digits is 4.

Solution 17. Note that number of eight-digit numbers with sum of digits equal to 4 is same as the number of solutions of

$$x_1 + x_2 + \cdots + x_8 = 4$$

with $x_1 \geq 1$ and $0 \leq x_1, \ldots, x_8 \leq 9$. It is also equal to the number of solutions of

$$y_1 + y_2 + \cdots + y_8 = 3$$

in nonnegative integers, which is $\binom{3+8-1}{8-1} = \binom{10}{7} = 120$. ■

Example 1.18 (India RMO 2014e P4)**.** A person moves in the x-y plane moving along points with integer co-ordinates x and y only. When she is at a point (x, y), she takes a step based on the following rules:

(a) if $x + y$ is even she moves to either $(x + 1, y)$ or $(x + 1, y + 1)$,

(b) if $x + y$ is odd she moves to either $(x, y + 1)$ or $(x + 1, y + 1)$.

How many distinct paths can she take to go from $(0, 0)$ to $(8, 8)$ given that she took exactly three steps to the right ((x, y) to $(x + 1, y)$)?

Solution 18. If R, U, D denote the number of steps taken to the right, upwards and along the diagonal, then the coordinate of the final point is $(R+D, U+D)$. Since R is equal to 3, it follows that D is equal to 5 and U is equal to 3. Note that between two consecutive steps to the right, there is an odd number of upward steps. So between two consecutive steps to the right, there is exactly one upward step. Thus the moves to the right and the upward moves form the sequence $RURURU$ or $URURUR$. Since the sum of the coordinates of the initial point is even and a diagonal move does change the parity of this sum, a move to the right is taken before the first upward move. So the moves to the right and the upward moves form the sequence $RURURU$. Hence given a path from $(0, 0)$ to $(8, 8)$, the five diagonal moves are placed in the blanks below (a blank space might contain no diagonal move or more than one diagonal move).

Conversely, any filling of the following blanks with five D's give us a path from $(0,0)$ to $(8,8)$.

$$-R-U-R-U-R-U-$$

So the number of distinct paths is equal to the number of solutions of the equation

$$x_1+x_2+\cdots+x_7=5$$

in nonnegative integers, which is equal to $\binom{5+7-1}{6}=\binom{11}{6}$. ■

Example 1.19. Let A, B be two finite sets. Denote by B^A the set of functions from $A \to B$. Show that B^A has cardinality $(\#B)^{\#A}$ when $A \neq \emptyset$.

Solution 19. Given an element a in A, its image can be any one of the $\#B$ elements of B. So there are $(\#B)^{\#A}$ functions from A to B. ■

Example 1.20. Determine the number of maps f from the set $\{1,2,3\}$ into the set $\{1,2,3,4,5\}$ such that $f(i)\leq f(j)$ whenever $i<j$.

Solution 20. The number of such maps is

$$\binom{5}{3}+2\binom{5}{2}+\binom{5}{1}=35.$$

■

Example 1.21 (India RMO 2015a P4, India RMO 2015b P4, India RMO 2015d P4, India RMO 2015e P4)**.** Suppose n objects are placed along a circle at equal distances with $n\in\{28,32,36,40\}$. In how many ways can 3 objects be chosen from among them so that no two of the three chosen objects are adjacent nor diametrically opposite?

Solution 21. Let m be an even number. Fix a regular polygon P with m vertices. Then three vertices can be chosen in $\binom{m}{3}$ ways. Note that exactly m of these choices contain two pairs of adjacent vertices, and exactly $m(m-4)$ of these choices contain only a single pair of adjacent vertices. This shows that three vertices of P can be chosen in $\binom{m}{3}-m-m(m-4)$ ways so that no two of the chosen vertices are adjacent. Note that among these type of choices of three vertices, exactly $\frac{m}{2}(m-6)$ contain a pair of diametrically opposite vertices. So the required number is equal to

$$\begin{aligned}&\binom{m}{3}-m-m(m-4)-\frac{m}{2}(m-6)\\&=\binom{m}{3}-m(m-3)-\frac{m}{2}(m-6)\\&=\frac{m}{6}(m^2-12m+38).\end{aligned}$$

■

Example 1.22 (India RMO 2015c P2)**.** Determine the number of 3 digit numbers in base 10 having at least one digit equal to 5 and at most one digit equal to 3.

Solution 22. Note that the number of three digit numbers satisfying the given conditions is equal to the number ordered triples (a, b, c) such that $a \neq 0$, $0 \leq a, b, c \leq 9$, at least one of a, b, c is equal to 5 and at most one of them is equal to 3. The number of triples (a, b, c) with $a \neq 0$, $0 \leq a, b, c \leq 9$ is equal to $9 \times 10 \times 10 = 900$. Note that the number of triples (a, b, c) such that $a \neq 0$, $a, b, c \in \{0, 1, 2, 3, 4, 6, 7, 8, 9\}$ is equal to $8 \times 9 \times 9 = 9 \times 72$. So there are $900 - 9 \times 72 = 9 \times 28 = 252$ triples (a, b, c) such that $a \neq 0$, $0 \leq a, b, c \leq 9$, at least one of a, b, c is equal to 5. Note that among such triples, there are three triples which contain more than one 3 (namely, $(5, 3, 3), (3, 5, 3), (3, 3, 5)$). So the required number is $252 - 3 = 249$. ■

Solution 23. Let A_0 denote the number of three digit numbers with no digit equal to 3, let A_1 denote the number of three digit numbers with exactly one digit equal to 3, and let B denote the number of three digit numbers with no digit equal to 5. We need to determine the size of the set $B^c \cap (A_0 \cup A_1)$, where B^c denotes the complement of B in the set of three digit numbers. Note that

$$\begin{aligned} |B^c \cap (A_0 \cup A_1)| &= |A_0 \cup A_1| - |(A_0 \cup A_1) \cap B| \\ &= |A_0| + |A_1| - |A_0 \cap B| - |A_1 \cap B|. \end{aligned}$$

Observe that

$$\begin{aligned} |A_0| &= 8 \cdot 9 \cdot 9, \\ |A_1| &= 9 \cdot 9 + 8 \cdot 9 + 8 \cdot 9, \end{aligned}$$

and similarly,

$$\begin{aligned} |A_0 \cap B| &= 7 \cdot 8 \cdot 8, \\ |A_1 \cap B| &= 8 \cdot 8 + 7 \cdot 8 + 7 \cdot 8. \end{aligned}$$

This yields

$$\begin{aligned} |B^c \cap (A_0 \cup A_1)| &= 9 \cdot 9 \cdot 9 + 2 \cdot 8 \cdot 9 - 8 \cdot 8 \cdot 8 - 2 \cdot 7 \cdot 8 \\ &= 2 \cdot 2 \cdot 8 + 81 + 72 + 64 \\ &= 249. \end{aligned}$$

■

Example 1.23 (India RMO 2016a P4)**.** Find the number of all 6-digit natural numbers having exactly three odd digits and three even digits.

Solution 24. The required number is equal to

$$\begin{aligned}&\binom{6}{3} \times 5^3 \times 5^3 - \binom{5}{3} \times 5^3 \times 5^2\\ &= 20 \times 5^6 - 10 \times 5^5\\ &= 5^6(20-2)\\ &= 281250.\end{aligned}$$

■

Example 1.24 (India RMO 2016b P4)**.** How many 6-digit natural numbers containing only the digits $1, 2, 3$ are there in which 3 occurs exactly twice and the number is divisible by 9?

Solution 25. Note that

$$\begin{aligned}1+1+1+1+3+3 &\equiv 1 \pmod 9,\\ 1+1+1+2+3+3 &\equiv 2 \pmod 9,\\ 1+1+2+2+3+3 &\equiv 3 \pmod 9,\\ 1+2+2+2+3+3 &\equiv 4 \pmod 9,\\ 2+2+2+2+3+3 &\equiv 5 \pmod 9\end{aligned}$$

hold. This shows that 9 does not divide the sum of $a, b, c, d, 3, 3$ for any choice of $a, b, c, d \in \{1, 2\}$. Hence, there are no 6-digit natural numbers satisfying the given conditions. ■

Example 1.25. [Sob13, Exercise 5.1] Find the number of surjective functions $\{1, 2, \dots, n+1\} \to \{1, 2, \dots, n\}$.

Solution 26. To arrange $n+1$ elements in n slots keeping no vacant slot, we could first choose two elements from $n+1$ elements and treat these two elements as a pair. Then arrange this pair along with the remaining $n-1$ elements in n slots. Note that all the required arrangements can be obtained in this way. So there $\binom{n+1}{2} \times n!$ surjective functions.

Alternatively, note that for each solution of the equation $x_1 + \cdots + x_n = n+1$ in positive integers, we could choose two elements from $\{1, 2, \dots, n+1\}$ and permute the rest. So there are $n \times \binom{n+1}{2} \times (n-1)! = \binom{n+1}{2} \times n!$ surjective functions. ■

Example 1.26 (India RMO 2016f P2)**.** At an international event there are 100 countries participating, each with its own flag. There are 10 distinct flagpoles at the stadium, labelled #1, #2, …, #10 in a row. In how many ways can all the 100 flags be hoisted on these 10 flagpoles, such that for each i from 1 to 10, the flagpole #i has at least i flags? (Note that the vertical order of the flagpoles on each flag is important)

Solution 27. Let us determine the number of solutions of

$$x_1 + \cdots + x_{10} = 100$$

in the positive integers satisfying $x_i \geq i$ for all $1 \leq i \leq 10$. Note that the solutions are in one-to-one correspondence with the solutions of

$$y_1 + \cdots + y_{10} = 100 - (1 + 2 + \cdots + 9)$$

in the positive integers. Hence, there are $\binom{54}{9}$ solutions of

$$x_1 + \cdots + x_{10} = 100$$

in the positive integers satisfying $x_i \geq i$ for all $1 \leq i \leq 10$. Note that corresponding to each such solution there are 100! arrangements of the flags. Hence the flags can be hoisted in $100! \times \binom{54}{9}$ ways. ∎

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