Arrange in order

MOPSS

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Mathematics Olympiad

Problem Solving Sessions

MOPSS

Department of Mathematics
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https://jpsaha.github.io/MOTP/MOPSS/

Suggested readings

- Evan Chen's advice On reading solutions, available at https://blog.evanchen.cc/2017/03/06/on-reading-solutions/.
- Evan Chen's Advice for writing proofs/Remarks on English, available at https://web.evanchen.cc/handouts/english/english.pdf.
- Notes on proofs by Evan Chen from OTIS Excerpts [**Che25**, Chapter 1].
- Tips for writing up solutions by Edward Barbeau, available at https://www.math.utoronto.ca/barbeau/writingup.pdf.
- Evan Chen discusses why math olympiads are a valuable experience for high schoolers in the post on Lessons from math olympiads, available at https://blog.evanchen.cc/2018/01/05/lessons-from-math-olympiads/.

List of problems and examples

§1 Arrange in Order

Example 1.1. Consider the squares of an 8×8 chessboard filled with the numbers 1 to 64 as in the figure below. If we choose 8 squares with the property that there is exactly one from each row and exactly one from each column, and add up the numbers in the chosen squares, show that the sum obtained is always 260.

1	2	3	4	5	6	7	8
9	10	11	12	13	14	15	16
17	18	19	20	21	22	23	24
25	26	27	28	29	30	31	32
33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48
49	50	51	52	53	54	55	56
57	58	59	60	61	62	63	64

Solution 1. Denote by a_{ij} the entry common to the i-th row and j-th column. Note that choosing 8 squares as stated would be same as choosing 8 numbers $a_{i_1 j_1}, \ldots, a_{i_8 j_8}$ such that $i_1, \ldots, i_8$ are equal to $1, \ldots, 8$ in some order and so are $j_1, \ldots, j_8$. Reordering $i_1, \ldots, i_8$ if necessary, we assume that $i_1 = 1, \ldots, i_8 = 8$. Note that a_{ij} is equal to $8(i-1)+j$ for $1 \le i, j \le 8$. Hence

$$
\begin{aligned}
a_{i_1 j_1} + \cdots + a_{i_8 j_8} &= 8(1-1) + j_1 + 8(2-1) + j_2 + \cdots + 8(8-1) + j_8 \\
&= 8(1+2+\cdots+7) + (j_1 + \cdots + j_8) \\
&= 8 \times 4 \times 7 + (1+2+\ldots 8) \\
&= 224 + 36 \\
&= 260.
\end{aligned}
$$

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Example 1.2 (India RMO 2002 P4)**.** Suppose the integers $1, 2, 3, \ldots, 10$ are split into two disjoint collections a_1, a_2, a_3, a_4, a_5 and b_1, b_2, b_3, b_4, b_5 such that $a_1 < a_2 < a_3 < a_4 < a_5$ and $b_1 > b_2 > b_3 > b_4 > b_5$.

(i) Show that the larger number in any pair $\{a_j, b_j\}$, $1 \le j \le 5$, is at least 6.

(ii) Show that $|a_1 - b_1| + |a_2 - b_2| + |a_3 - b_3| + |a_4 - b_4| + |a_5 - b_5| = 25$ for every such partition.

Solution 2. Let $1 \leq j \leq 5$ be an integer. Note that the inequalities

$$a_1 < a_2 < \cdots < a_j, \quad b_j > b_{j+1} > \cdots > b_5$$

hold. Since $a_1, a_2, \ldots, a_j, b_j, b_{j+1}, \ldots, b_5$ are six distincts elements of the set $\{1, 2, \ldots, 10\}$, their maximum is at least 6. The above inequalities show that this maximum is equal to $\max\{a_j, b_j\}$. This proves part (i).

Note that the sets $\{a_1, b_1\}, \{a_2, b_2\}, \{a_3, b_3\}, \{a_4, b_4\}, \{a_5, b_5\}$ are pairwise disjoint subsets of $\{1, 2, \ldots, 10\}$, and the maximum of any of these sets is at least 6. Consequently, the maximums of these sets are equal to $6, 7, 8, 9, 10$ in some order. Thus their minimums are equal to $1, 2, 3, 4, 5$ in some order. Hence, the integer

$$|a_1 - b_1| + |a_2 - b_2| + |a_3 - b_3| + |a_4 - b_4| + |a_5 - b_5|$$

is equal to the difference of the sum of of the maximums of these sets and the sum of the minimums of these sets, which is equal to

$$6 + 7 + 8 + 9 + 10 - (1 + 2 + 3 + 4 + 5) = 25.$$

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Example 1.3 (India RMO 2006 P4)**.** A 6×6 square is dissected into 9 rectangles by lines parallel to its sides such that all these rectangles have only integer sides. Prove that there are always two congruent rectangles.

Solution 3. On the contrary, let us assume that it is possible to divide a 6×6 square into 9 disjoint subrectangles with each side having integer length, and parallel to one of the sides of the square such that no two subrectangles are congruent. Denote the areas of these subrectanlges by

$$a_1 \leq a_2 \leq a_3 \leq a_4 \leq a_5 \leq a_6 \leq a_7 \leq a_8 \leq a_9.$$

Note that if for some $1 \leq i \leq 9$, the integer a_i is a prime or equal to 1, then a_i is the area of exactly one of these 9 subrectangles. Noting that

$$4 = 1 \times 4 = 2 \times 2, \quad 6 = 1 \times 6, 2 \times 3$$

are the only factorizations of 4 and 6 over the positive integers upto reordering, it follows that if for some $1 \leq i \leq 9$, the integer a_i is equal to one of $4, 6, 8$, then it can be the area of at most two such subrectangles. Moreover, note that for no integer $1 \leq i \leq 9$, a_i is equal to 7 since these subrectangles are contained in a 6×6 square. It follows that

$$a_1 \geq 1, a_2 \geq 2, a_3 \geq 3, a_4 \geq 4, a_5 \geq 4, a_6 \geq 5, a_7 \geq 6, a_8 \geq 6, a_9 \geq 8.$$

So the total area of these nine subrectangles is at least

$$1 + 2 + 3 + 4 + 4 + 5 + 6 + 6 + 8 \geq 39,$$

which is larger than 36. This proves the given statement. ■

References

[**Che25**] Evan Chen. *The OTIS Excerpts*. Available at https://web.evanchen.cc/excerpts.html. 2025, pp. vi+289 (cited p. 1)