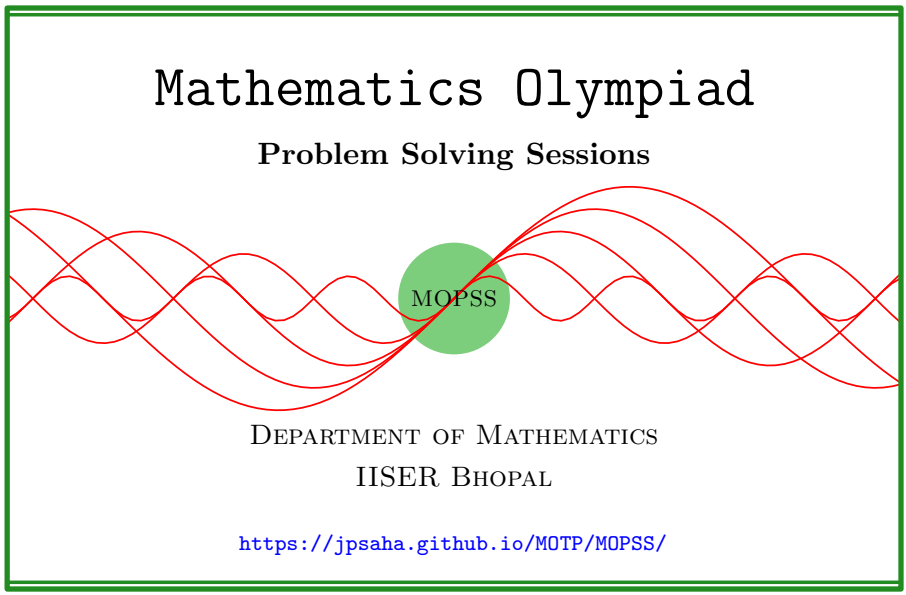
$$a^3 + b^3 + c^3 - 3abc$$

MOPSS



Suggested readings

- Evan Chen's advice On reading solutions, available at https://blog.evanchen.cc/2017/03/06/on-reading-solutions/.
- Evan Chen's Advice for writing proofs/Remarks on English, available at https://web.evanchen.cc/handouts/english/english.pdf.
- Notes on proofs by Evan Chen from OTIS Excerpts [**Che25**, Chapter 1].
- Tips for writing up solutions by Edward Barbeau, available at https://www.math.utoronto.ca/barbeau/writingup.pdf.
- Evan Chen discusses why math olympiads are a valuable experience for high schoolers in the post on Lessons from math olympiads, available at https://blog.evanchen.cc/2018/01/05/lessons-from-math-olympiads/.

List of problems and examples

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§1 $a^3+b^3+c^3-3abc$

Example 1. Let a, b, c be real numbers. Show that

$$\begin{aligned} a^3+b^3+c^3-3abc &= (a+b+c)(a^2+b^2+c^2-ab-bc-ca) \\ &= (a+b+c)\left((a+b+c)^2-3(ab+bc+ca)\right) \\ &= \frac{1}{2}(a+b+c)\left((a-b)^2+(b-c)^2+(c-a)^2\right). \end{aligned}$$

Remark. An immediate approach would be to begin from the expression $(a+b+c)(a^2+b^2+c^2-ab-bc-ca)$ at RHS (the right-hand side), multiply it out and the cancellations would lead to the expression $a^3+b^3+c^3-3abc$. This would definitely provide a proof of the above. However, there is another way to argue as below.

Solution 1. Observe that

$$\begin{aligned} &a^2+b^2+c^2-ab-bc-ca \\ &= a^2+b^2+c^2+2ab+2bc+2ca-3(ab+bc+ca) \\ &= (a+b+c)^2-3(ab+bc+ca), \\ &2(a^2+b^2+c^2-ab-bc-ca) \\ &= a^2-2ab+b^2+b^2-2bc+c^2+c^2-2ca+a^2 \\ &= (a-b)^2+(b-c)^2+(c-a)^2. \end{aligned}$$

Note that

$$\begin{aligned} &a^3+b^3+c^3-3abc \\ &= (a+b)^3-3ab(a+b)+c^3-3abc \\ &= (a+b)^3+c^3-3ab(a+b)-3abc \\ &= (a+b)^3+c^3-3ab(a+b+c) \end{aligned}$$

The content posted here and at this blog by Evan Chen are quite useful.

$$
\begin{aligned}
&= (a+b+c)^3 - 3(a+b)c(a+b+c) - 3ab(a+b+c) \\
&= (a+b+c)((a+b+c)^2 - 3(a+b)c - 3ab) \\
&= (a+b+c)(a^2+b^2+c^2+2ab+2bc+2ca-3ab-3bc-3ca) \\
&= (a+b+c)(a^2+b^2+c^2-ab-bc-ca).
\end{aligned}
$$

■

Remark. There is another way to prove the above identity.

Solution 2. Consider the polynomial

$$P(X) = X^3 - (a+b+c)X^2 + (ab+bc+ca)X - abc.$$

Since a, b, c are the roots[1] of the equation $P(X) = 0$, we obtain

$$
\begin{aligned}
a^3 - (a+b+c)a^2 + (ab+bc+ca)a - abc &= 0, \\
b^3 - (a+b+c)b^2 + (ab+bc+ca)b - abc &= 0, \\
c^3 - (a+b+c)c^2 + (ab+bc+ca)c - abc &= 0.
\end{aligned}
$$

Adding them yields

$$a^3+b^3+c^3 - (a+b+c)(a^2+b^2+c^2) + (ab+bc+ca)(a+b+c) - 3abc = 0.$$

This proves that

$$a^3+b^3+c^3 - 3abc = (a+b+c)(a^2+b^2+c^2-ab-bc-ca).$$

■

The above identity has the following immediate consequence.

Corollary

If a, b, c are real numbers satisfying $a+b+c=0$, then

$$a^3+b^3+c^3 = 3abc.$$

Exercise 2 (Moscow Mathematical Olympiad First Round 1940 Grades 7–8 P1)**.** Factor $(x-y)^3 + (y-z)^3 + (z-x)^3$.

[1] If it is not clear, then the following equalities may directly be verified.

Walkthrough —
(a)

Solution 3. Note that if $a+b+c=0$, then $a^3+b^3+c^3=3abc$. This gives

$$(x-y)^3+(y-z)^3+(z-x)^3=3(x-y)(y-z)(z-x).$$

■

Remark. The following proof is direct, and of course, it works.

$$\begin{aligned}
&(x-y)^3+(y-z)^3+(z-x)^3\\
&=x^3-3x^2y+3xy^2-y^3\\
&+y^3-3y^2z+3yz^2-z^3\\
&+z^3-3z^2x+3zx^2-x^3\\
&=-3x^2y+3xy^2-3y^2z+3yz^2-3z^2x+3zx^2\\
&=-3xy(x-y)-3y^2z+3yz^2-3z^2x+3zx^2\\
&=-3xy(x-y)-3y^2z+3zx^2+3yz^2-3z^2x\\
&=-3xy(x-y)+3z(x^2-y^2)-3z^2(x-y)\\
&=-3xy(x-y)+3z(x-y)(x+y)-3z^2(x-y)\\
&=3(x-y)\big(-xy+z(x+y)-z^2\big)\\
&=3(x-y)\big(-xy+zx+zy-z^2\big)\\
&=3(x-y)\big(-x(y-z)+z(y-z)\big)\\
&=3(x-y)(y-z)(z-x).
\end{aligned}$$

However, the former solution is less cumbersome, and more elegant.

Exercise 3 (RMO 2002 P2,). Solve the following equation for real x:

$$(x^2+x-2)^3+(2x^2-x-1)^3=27(x^2-1)^3.$$

Solution 4. The given equation is equivalent to

$$(x^2+x-2)^3+(2x^2-x-1)^3+(-3x^2+3)^3=0.$$

Note that $x^2+x-2, 2x^2-x-1, -3x^2+3$ add up to zero. This implies

$$\begin{aligned}
&(x^2+x-2)^3+(2x^2-x-1)^3+(-3x^2+3)^3\\
&=3(x^2+x-2)(2x^2-x-1)(-3x^2+3)\\
&=-9(x+2)(x-1)(x-1)(2x-1)(x-1)(x+1).
\end{aligned}$$

The content posted here and at this blog by Evan Chen are quite useful.

Thus the required solutions for x are

$$-2, -1, \frac{1}{2}, 1.$$

■

Exercise 4 (Formula of Unity/The Third Millennium 2023/2024 Qualifying Round Grade R11 P3, S. Pavlov)**.** Let a, b, c be nonzero real numbers such that

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} = 6, \quad \frac{b}{a} + \frac{c}{b} + \frac{a}{c} = 2.$$

What could be the value of the expression

$$\frac{a^3}{b^3} + \frac{b^3}{c^3} + \frac{c^3}{a^3}?$$

> **Walkthrough** —
> **(a)**

Solution 5. Write $x = \frac{a}{b}, y = \frac{b}{c}, z = \frac{c}{a}$. Note that

$$x + y + z = 6, \quad xy + yz + zx = 2.$$

This yields

$$\begin{aligned} x^3 + y^3 + z^3 &= 3 + (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx) \\ &= 3 + (x + y + z)\left((x + y + z)^2 - 3(xy + yz + zx)\right) \\ &= 3 + 6 \times (6^2 - 3 \cdot 2) \\ &= 183. \end{aligned}$$

■

Exercise 5 (Formula of Unity/The Third Millennium 2022/2023 Qualifying Round Grade R11 P5)**.** Find all real a, b, c such that

$$27^{a^2+b+c+1} + 27^{b^2+c+a+1} + 27^{c^2+a+b+1} = 3.$$

> **Walkthrough** —
> **(a)**

Solution 6. For any three real numbers a, b and c, note that

$$\begin{aligned}
&27^{a^2+b+c+1} + 27^{b^2+c+a+1} + 27^{c^2+a+b+1} \\
&\geq 3 \cdot 3^{a^2+b+c+1} \cdot 3^{b^2+c+a+1} \cdot 3^{c^2+a+b+1} \\
&\qquad \text{(using Example 1 and that } 3^x \geq 0 \text{ for any real number } x\text{)} \\
&= 3 \cdot 3^{a^2+b^2+c^2+2a+2b+2c+3} \\
&= 3 \cdot 3^{(a+1)^2+(b+1)^2+(c+1)^2}
\end{aligned}$$

hold. This shows that if a, b, c are real numbers satisfying the given condition, then

$$a = b = c = -1.$$

Moreover, note that for $a = b = c = -1$, the equality

$$27^{a^2+b+c+1} + 27^{b^2+c+a+1} + 27^{c^2+a+b+1} = 3$$

holds. Hence, the solution of the given equation is

$$a = b = c = -1.$$

■

Exercise 6 (INMO 2002 P2,). Find the smallest positive value taken by $a^3 + b^3 + c^3 - 3abc$ for positive integers a, b, c. Find all a, b, c which give the smallest value.

Walkthrough —

(a) Note that $a = b = c = 1$ won't work, not even taking all of a, b, c to be equal would be of any use. In other words, at least two of a, b, c have to be unequal.

(b) By taking $a = 1, b = 2, c = 1$, one can find that $a^3 + b^3 + c^3 - 3abc = 4$. Next, we need determine whether $a^3 + b^3 + c^3 - 3abc$ can be equal to $1, 2, 3$ or 4 for positive integers a, b, c.

(c) Use

$$\begin{aligned}
a^3 + b^3 + c^3 - 3abc &= (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca) \\
&= \frac{1}{2}(a + b + c)((a - b)^2 + (b - c)^2 + (c - a)^2)
\end{aligned}$$

to get a lower bound on $a^3 + b^3 + c^3 - 3abc$.

Solution 7. Let a, b, c be positive integers such that $a^3 + b^3 + c^3 - 3abc$ is positive. Note that they cannot be equal, and hence at least two of them are

distinct. Since $a^3+b^3+c^3-3abc$ is symmetric[2] in a,b,c, we may assume[3] that $a \neq b$.

Apart from the integers a and b, there is another pair of two integers among a,b,c which are not equal, that is, $b \neq c$ or $c \neq a$ holds. Indeed, if both of these two inequalities fail to hold, then $b=c$ and $c=a$ hold, and then we would have $a=b$, which is a contradiction. Note that

$$\begin{aligned}
&a^3+b^3+c^3-3abc \\
&= (a+b+c)(a^2+b^2+c^2-ab-bc-ca) \\
&= \frac{1}{2}(a+b+c)\left((a-b)^2+(b-c)^2+(c-a)^2\right) \\
&\geq \frac{1}{2}(a+b+c)(1^2+1^2) \\
&\qquad \text{(since at least two of } a-b, b-c, c-a \text{ are nonzero, and } a+b+c>0) \\
&\geq a+b+c \\
&\geq 1+2+1 \quad \text{(since at least two of } a-b, b-c, c-a \text{ are nonzero, and } a,b,c \geq 1) \\
&= 4.
\end{aligned}$$

Also note that if $c>1$, then

$$a^3+b^3+c^3-3abc>4.$$

For $a=1, b=2, c=1$, we obtain

$$a^3+b^3+c^3-3abc=4.$$

Hence, the smallest positive value taken by $a^3+b^3+c^3-3abc$, for positive integers a,b,c, is equal to 4.

Moreover, if a,b,c are positive integers such that $a^3+b^3+c^3-3abc$ takes the value 4, then at least two of a,b,c are unequal, and the above argument shows that

$$a+b+c \leq a^3+b^3+c^3-3abc \leq 4,$$

and consequently, two of a,b,c are equal to 1 and the remaining one is equal to 2. Hence, $a^3+b^3+c^3-3abc$ takes the value 4 precisely when

$$(a,b,c)=(1,1,2),(1,2,1),(2,1,1).$$

■

For more exercises around this theme, we refer to [**AE11**, §1.1].

[2] A reader unfamiliar with this term may require to look online.
[3] How we may do so? It does require a thought.

References

[**AE11**] Titu Andreescu and Bogdan Enescu. *Mathematical Olympiad treasures*. Second. Birkhäuser/Springer, New York, 2011, pp. viii+253. ISBN: 978-0-8176-8252-1; 978-0-8176-8253-8 (cited p. 7)

[**Che25**] Evan Chen. *The OTIS Excerpts*. Available at https://web.evanchen.cc/excerpts.html. 2025, pp. vi+289 (cited p. 1)