Warm up

MOPSS

8 May 2025

Mathematics Olympiad

Problem Solving Sessions

Department of Mathematics
IISER Bhopal

https://jpsaha.github.io/MOTP/MOPSS/

Suggested readings

- Evan Chen's advice On reading solutions, available at https://blog.evanchen.cc/2017/03/06/on-reading-solutions/.
- Evan Chen's Advice for writing proofs/Remarks on English, available at https://web.evanchen.cc/handouts/english/english.pdf.
- Notes on proofs by Evan Chen from OTIS Excerpts [**Che25**, Chapter 1].
- Tips for writing up solutions by Edward Barbeau, available at https://www.math.utoronto.ca/barbeau/writingup.pdf.
- Evan Chen discusses why math olympiads are a valuable experience for high schoolers in the post on Lessons from math olympiads, available at https://blog.evanchen.cc/2018/01/05/lessons-from-math-olympiads/.

List of problems and examples

1.1 Example (G. Galperin, Tournament of Towns, Autumn 1989, Junior, O Level, P4) . . . 2
1.2 Example (IMO 1959 P1, proposed by Poland) . . . 2
1.3 Example (India RMO 2019a P4) . . . 3
1.4 Example (Tournament of Towns, Spring 2014, Senior, A Level, P4 by I. I. Bogdanov) . . . 4
1.5 Example (Tournament of Towns, Spring 2020, Junior, O Level, P4 by Alexandr Yuran) . . . 4
1.6 Example (India RMO 2015e P3) . . . 4
1.7 Example (India RMO 2016e P5) . . . 6

§1 Warm up

It would be good to go through [**Che25**, Chapter 1, Notes on proofs].

Example 1.1 (G. Galperin, Tournament of Towns, Autumn 1989, Junior, O Level, P4)**.** Find the solutions of the equation

$$x + \frac{1}{y + \frac{1}{z}} = \frac{10}{7} \tag{1}$$

in positive integers.

Solution 1. Let x, y, z be positive integers satisfying Eq. (1). Since $y \geq 1$ and $\frac{1}{z} > 0$, it follows that $y + \frac{1}{z} > 1$, which gives $0 < \frac{1}{y+\frac{1}{z}} < 1$. Using Eq. (1), it follows that $x = 1$, and hence $y + \frac{1}{z} = \frac{7}{3}$. By a similar argument as above[1], it follows that $y = 2$ and consequently, $z = 3$.

Moreover, for $x = 1, y = 2, z = 3$, Eq. (1) holds.

This proves that $x = 1, y = 2, z = 3$ is the only solution[2] of Eq. (1). ∎

> **Remark.** Is the statement that **for $x = 1, y = 2, z = 3$, Eq. (1) holds** in the above argument **redundant**? Or, is it not so? Think about it. Further, it would be worth going through [**Che25**, Chapter 1].

Example 1.2 (IMO 1959 P1, proposed by Poland)**.** Prove that the fraction

$$\frac{21n + 4}{14n + 3}$$

is irreducible for every natural number n.

[1] Write the argument instead of resorting to using "by a similar argument" unless it is clear to you. Even then, consider it as an exercise and write it down!

[2] It means $x = 1, y = 2, z = 3$ is **a solution** to Eq. (1), and that it is the **only solution**, i.e. if we are given a solution, it cannot be different from $x = 1, y = 2, z = 3$. Does the above argument prove both?

We need to show that $21n+4, 14n+3$ have no factor in common other than 1 for every natural number n.

Summary — It follows from considering the greatest common divisor of the numerator and the denominator.

Walkthrough —

- The summand $21n$ from the numerator and the summand $14n$ from the denominator do not "balance well".
- One way "enforce balancing" would be to consider

$$2 \cdot 21n - 3 \cdot 14n,$$

which vanishes.
- Does the above "ad hoc thoughts" help to conclude?

Solution 2. Let n be a natural number. It is enough to show that the greatest common divisor of the integers $21n+4, 14n+3$ is equal to 1. Note that any common divisor of $21n+4, 14n+3$ divides

$$2(21n+4) - 3(14n+3) = -1.$$

This shows that the greatest common divisor of the integers $21n+4, 14n+3$ is equal to 1, completing the proof. ■

Example 1.3 (India RMO 2019a P4)**.** Consider the following 3×2 array formed by using the numbers $1, 2, 3, 4, 5, 6$,

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix} = \begin{pmatrix} 1 & 6 \\ 2 & 5 \\ 3 & 4 \end{pmatrix}.$$

Observe that all row sums are equal, but the sum of the square of the squares is not the same for each row. Extend the above array to a $3 \times k$ array $(a_{ij})_{3\times k}$ for a suitable k, adding more columns, using the numbers $7, 8, 9, \dots, 3k$ such that

$$\sum_{j=1}^{k} a_{1j} = \sum_{j=1}^{k} a_{2j} = \sum_{j=1}^{k} a_{3j} \quad \text{and} \quad \sum_{j=1}^{k} (a_{1j})^2 = \sum_{j=1}^{k} (a_{2j})^2 = \sum_{j=1}^{k} (a_{3j})^2$$

Solution 3. Note that

$$\begin{pmatrix} 1 & 6 & 2+6 & 5+6 & 3+2\cdot 6 & 4+2\cdot 6 \\ 2 & 5 & 3+6 & 4+6 & 1+2\cdot 6 & 6+2\cdot 6 \\ 3 & 4 & 1+6 & 6+6 & 2+2\cdot 6 & 5+2\cdot 6 \end{pmatrix}$$

works. ■

Example 1.4 (Tournament of Towns, Spring 2014, Senior, A Level, P4 by I. I. Bogdanov)**.** The King called two wizards. He ordered First Wizard to write down 100 positive real numbers (not necessarily distinct) on cards without revealing them to Second Wizard. Second Wizard must correctly determine all these numbers, otherwise both wizards will lose their heads. First Wizard is allowed to provide Second Wizard with a list of distinct numbers, each of which is either one of the numbers on the cards or a sum of some of these numbers. He is not allowed to tell which numbers are on the cards and which numbers are their sums. Finally the King tears as many hairs from each wizard's beard as the number of numbers in the list given to Second Wizard. What is the minimal number of hairs each wizard should lose to stay alive?

Example 1.5 (Tournament of Towns, Spring 2020, Junior, O Level, P4 by Alexandr Yuran)**.** For some integer n, the equation $x^2+y^2+z^2-xy-yz-zx=n$ has an integer solution x, y, z. Prove that the equation $x^2+y^2-xy=n$ also has an integer solution x, y.

Walkthrough — Note that the identity

$$x^2+y^2+z^2-xy-yz-zx=(x-y)^2-(x-y)(z-y)+(z-y)^2$$

holds.

Example 1.6 (India RMO 2015e P3)**.** Find all fractions which can be written simultaneously in the forms

$$\frac{7k-5}{5k-3} \quad \text{and} \quad \frac{6\ell-1}{4\ell-3}$$

for some integers k, ℓ.

Solution 4. The solution relies on the following claim.

Claim — Suppose k, ℓ are integers. Then the equality

$$\frac{7k-5}{5k-3}=\frac{6\ell-1}{4\ell-3}$$

is equivalent to the pair (k, ℓ) being equal to one of

$$(0,6),(1,-1),(6,-6),(13,-7),(-2,-22),(-3,-15),(-8,-10),(-15,-9). \tag{2}$$

Proof of the Claim. Suppose k, ℓ are integers. Observing that $5k-3$ and $4\ell-3$ are nonzero, it follows that

$$\frac{7k-5}{5k-3}=\frac{6\ell-1}{4\ell-3}$$

$$
\begin{aligned}
\iff \quad & (7k-5)(4\ell-3) = (5k-3)(6\ell-1) \\
\iff \quad & 28k\ell - 20\ell - 21k + 15 = 30k\ell - 18\ell - 5k + 3 \\
\iff \quad & 2k\ell + 2\ell + 16k - 12 = 0 \\
\iff \quad & k\ell + \ell + 8k - 6 = 0 \\
\iff \quad & (k+1)(\ell+8) = 14.
\end{aligned}
$$

This implies that $k+1$ is equal to

$$\pm 1, \pm 2, \pm 7, \pm 14,$$

i.e. k is equal to

$$0, 1, 6, 13, -2, -3, -8, -15. \tag{3}$$

It follows that

$$(k+1)(\ell+8) = 14$$

is equivalent to (k, ℓ) being equal to one of the pairs as in Eq. (2). This proves the Claim. □

Note that if a fraction can be written simultaneously in the forms

$$\frac{7k-5}{5k-3} \quad \text{and} \quad \frac{6\ell-1}{4\ell-3}$$

for two integers k, ℓ, then the Claim implies that (k, ℓ) is equal to the pairs as in Eq. (2), and then k is equal to the integers as in Eq. (3), and consequently, the fraction $\frac{7k-5}{5k-3}$, which is equal to $\frac{6\ell-1}{4\ell-3}$ (by the Claim again), is also equal to

$$\frac{5}{3}, 1, \frac{37}{27}, \frac{43}{31}, \frac{19}{13}, \frac{13}{9}, \frac{61}{43}, \frac{30}{19}. \tag{4}$$

Further[3], observe that the preceeding argument also proves that these fractions can be written simultaneously in the forms as stated above. Indeed, if (k, ℓ) is one of the pairs as in Eq. (2), and then k is equal to the integers as in Eq. (3), and consequently, the fraction $\frac{7k-5}{5k-3}$, which is equal to $\frac{6\ell-1}{4\ell-3}$ (by the Claim), is also equal to the fractions as in Eq. (4).

We conclude that the fractions as in Eq. (4) are precisely all the fractions with the required property. ■

[3]Note that the argument needs to go on since what we have proved so far does not complete the solution. The previous step only says that if a fraction can be written simultaneously in the forms as stated above (and **a priori, it is not clear if there is even a single fraction that can be expressed simultaneously in the stated forms**), then the fraction cannot be anything other than

$$\frac{5}{3}, 1, \frac{37}{27}, \frac{43}{31}, \frac{19}{13}, \frac{13}{9}, \frac{61}{43}, \frac{30}{19}.$$

This does not guarantee if any of these fractions enjoy the stated property.

If this causes any confusion, then it would be a good idea to go through [**Che25**, Chapter 1].

Example 1.7 (India RMO 2016e P5).

(i) A 7-tuple $(a_1, a_2, a_3, a_4, b_1, b_2, b_3)$ of pairwise distinct positive integers with no common factor is called a shy tuple if

$$a_1^2 + a_2^2 + a_3^2 + a_4^2 = b_1^2 + b_2^2 + b_3^2$$

and for all $1 \leq i < j \leq 4$ and $1 \leq k \leq 3$, $a_i^2 + a_j^2 \neq b_k^2$. Prove that there exist infinitely many shy tuples.

(ii) Show that 2016 can be written as a sum of squares of four distinct natural numbers.

Solution 5. For any integer $n \geq 2$,

$$(3, 4, 3n, 4n, 3(n^2 - 1), 6n, 4(n^2 + 1))$$

is a 7-tuple of pairwise distinct positive integers, and these integers have no common factor. This proves pari (i).

Note that $2016 = 2^5 \cdot 3^2 \cdot 7$ holds. Observe that $2 \cdot 3^2 \cdot 7$ can be expressed as $4^2 + 5^2 + 6^2 + 7^2$. This gives

$$2016 = 2^5 \cdot 3^2 \cdot 7 = 16^2 + 20^2 + 24^2 + 28^2,$$

expressing 2016 as the sum of the squares of four distinct natural numbers. ■

Remark. The identity

$$(x + y)^2 + (x - y)^2 = 2(x^2 + y^2)$$

can also be used to look for shy tuples. Note that

$$\begin{aligned}(x - 3y)^2 + (x - y)^2 + (x + y)^2 + (x + 3y)^2 &= 2(x^2 + 9y^2) + 2(x^2 + y^2)\\ &= (2x)^2 + 20y^2\\ &= (2x)^2 + (2y)^2 + (4y)^2.\end{aligned}$$

This leads to considering the tuple $(x - 3y, x - y, x + y, x - 3y, 2x, 2y, 4y)$. To make it a shy tuple, one can take x, y to be positive integers satisfying $x - 3y \geq 1$, and one can impose the condition that x, y are coprime, along with the additional condition that x, y are of different parity.

Alternatively, one can make use of the following identity.

$$(a + b + c)^2 + (-a + b + c)^2 + (a - b + c)^2 + (a + b - c)^2 = (2a)^2 + (2b)^2 + (2c)^2.$$

For more details, we refer to this AoPS thread.

References

[**Che25**] Evan Chen. *The OTIS Excerpts*. Available at https://web.evanchen.cc/excerpts.html. 2025, pp. vi+289 (cited pp. 1, 2, 5)