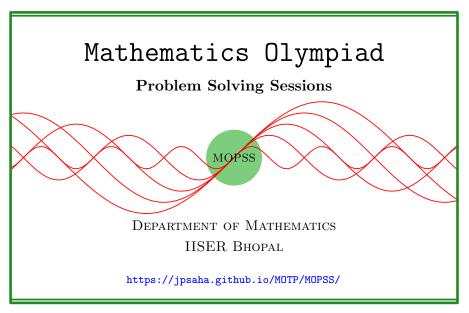
Viète's relations

MOPSS

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Suggested readings

- Evan Chen's advice On reading solutions, available at https://blog.evanchen.cc/2017/03/06/on-reading-solutions/.
- Evan Chen's Advice for writing proofs/Remarks on English, available at https://web.evanchen.cc/handouts/english/english.pdf.
- Notes on proofs by Evan Chen from OTIS Excerpts [Che25, Chapter 1].
- Tips for writing up solutions by Edward Barbeau, available at https://www.math.utoronto.ca/barbeau/writingup.pdf.
- Evan Chen discusses why math olympiads are a valuable experience for high schoolers in the post on Lessons from math olympiads, available at https://blog.evanchen.cc/2018/01/05/lessons-from-math-olympiads/.

List of problems and examples

§1 Viète's relations

Example 1.1 (India RMO 2012e P2). cf. [GA17, Problem 141] Let $P(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ be a polynomial of degree $n \geq 3$. Knowing that $a_{n-1} = -\binom{n}{1}$ and $a_{n-2} = \binom{n}{2}$, and that all the roots of P are real, find the remaining coefficients. Note that $\binom{n}{r} = \frac{n!}{(n-r)!r!}$.

Solution 1. Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ denote the roots of P. Note that

$$\begin{split} &\sum_{1 \leq i, j \leq n, i \neq j} (\alpha_i - \alpha_j)^2 \\ &= 2(n-1) \sum_{1 \leq i \leq n} \alpha_i^2 - 2 \sum_{1 \leq i, j \leq n, i \neq j} \alpha_i \alpha_j \\ &= 2(n-1) \sum_{1 \leq i \leq n} \alpha_i^2 - 4 \sum_{1 \leq i < j \leq n} \alpha_i \alpha_j \\ &= 2(n-1) \left(\sum_{1 \leq i \leq n} \alpha_i \right)^2 - 4(n-1) \left(\sum_{1 \leq i < j \leq n} \alpha_i \alpha_j \right) - 4 \sum_{1 \leq i < j \leq n} \alpha_i \alpha_j \\ &= 2n^2(n-1) - 4n \sum_{1 \leq i < j \leq n} \alpha_i \alpha_j \\ &= 2n^2(n-1) - 4n \binom{n}{2} \\ &= 0. \end{split}$$

Since $\alpha_1, \ldots, \alpha_n$ are real, it follows that they are all equal. Using $\alpha_1 + \cdots + \alpha_n = n$, we get

$$\alpha_1 = \alpha_2 = \dots = \alpha_n = 1.$$

This implies that $a_i = -\binom{n}{i}$ for any $0 \le i \le n-1$.

Solution 2. Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ denote the roots of P(x). Note that

$$(\alpha_1 - 1)^2 + (\alpha_2 - 1)^2 + \dots + (\alpha_n - 1)^2$$

$$= \alpha_1^2 + \alpha_2^2 + \dots + \alpha_n^2 + n - 2(\alpha_1 + \alpha_2 + \dots + \alpha_n)$$

$$= (\alpha_1 + \alpha_2 + \dots + \alpha_n)^2 - 2 \sum_{1 \le i < j \le n} \alpha_i \alpha_j + n - 2(\alpha_1 + \alpha_2 + \dots + \alpha_n)$$

$$= n^2 - n(n - 1) + n - 2n$$

$$= 0.$$

So the roots $\alpha_1, \alpha_2, \ldots, \alpha_n$ are all equal to 1. This implies that $P(x) = (x-1)^n$, and hence a_i is equal to $(-1)^i \binom{n}{i}$ for any $0 \le i \le n$.

References

- [Che25] EVAN CHEN. The OTIS Excerpts. Available at https://web.evanchen.cc/excerpts.html. 2025, pp. vi+289
- [GA17] RĂZVAN GELCA and TITU ANDREESCU. Putnam and beyond. Second. Springer, Cham, 2017, pp. xviii+850. ISBN: 978-3-319-58986-2; 978-3-319-58988-6. DOI: 10.1007/978-3-319-58988-6. URL: https://doi.org/10.1007/978-3-319-58988-6 (cited p. 2)