Roots of unity

MOPSS

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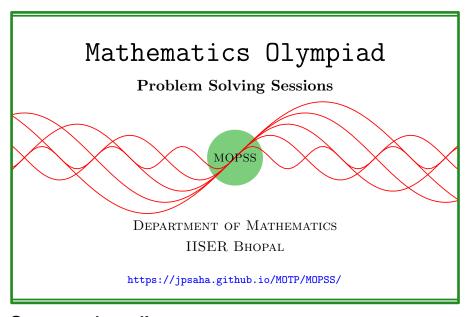# Mathematics Olympiad
Problem Solving Sessions

MOPSS

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Suggested readings

- Evan Chen's advice On reading solutions, available at `https://blog.evanchen.cc/2017/03/06/on-reading-solutions/`.

- Evan Chen's Advice for writing proofs/Remarks on English, available at `https://web.evanchen.cc/handouts/english/english.pdf`.

- Notes on proofs by Evan Chen from OTIS Excerpts [**Che25**, Chapter 1].

- Tips for writing up solutions by Edward Barbeau, available at `https://www.math.utoronto.ca/barbeau/writingup.pdf`.

- Evan Chen discusses why math olympiads are a valuable experience for high schoolers in the post on Lessons from math olympiads, available at `https://blog.evanchen.cc/2018/01/05/lessons-from-math-olympiads/`.

List of problems and examples

§1 Roots of unity

Some of the problems have been borrowed from Yufei Zhao's handout on polynomials.

Example 1.1 (India Pre-RMO 2012 P17)**.** Let x_1, x_2, x_3 be the roots of the equation $x^3 + 3x + 5 = 0$. What is the value of the expression

$$\left(x_1 + \frac{1}{x_1}\right)\left(x_2 + \frac{1}{x_2}\right)\left(x_3 + \frac{1}{x_3}\right)?$$

See also Example 1.2, USAMO 2014 P1.

Solution 1. Let $P(x)$ denote the polynomial $x^3 + 3x + 5$. Note that

$$\left(x_1 + \frac{1}{x_1}\right)\left(x_2 + \frac{1}{x_2}\right)\left(x_3 + \frac{1}{x_3}\right)$$

$$= \frac{1}{x_1 x_2 x_3}(x_1^2 + 1)(x_2^2 + 1)(x_3^2 + 1)$$

$$= \frac{1}{x_1 x_2 x_3}(x_1 + i)(x_2 + i)(x_3 + i)(x_1 - i)(x_2 - i)(x_3 - i)$$

$$= \frac{1}{x_1 x_2 x_3}P(-i)P(i)$$

$$= \frac{1}{-5}|P(i)|^2$$

$$= \frac{1}{-5}|5 - 2i|^2$$

$$= -\frac{29}{5}.$$

∎

Example 1.2 (USAMO 2014 P1)**.** Let a, b, c, d be real numbers such that $b - d \geq 5$ and all zeros x_1, x_2, x_3, x_4 of the polynomial $P(x) = x^4 + ax^3 + bx^2 + cx + d$ are real. Find the smallest value the product

$$(x_1^2 + 1)(x_2^2 + 1)(x_3^2 + 1)(x_4^2 + 1)$$

can take.

See also Example 1.1, India Pre-RMO 2012 P17.

Solution 2. Note that

$$(x_1^2 + 1)(x_2^2 + 1)(x_3^2 + 1)(x_4^2 + 1) = P(i)P(-i)$$
$$= (1 - b + d)^2 + (a - c)^2$$
$$= (b - d - 1)^2 + (a - c)^2$$
$$\geq 16.$$

Taking $a = c$ and $b = 5, d = 0$, we obtain

$$(x_1^2 + 1)(x_2^2 + 1)(x_3^2 + 1)(x_4^2 + 1) = 16.$$

Hence, the smallest value the product $(x_1^2 + 1)(x_2^2 + 1)(x_3^2 + 1)(x_4^2 + 1)$ can take is equal to 16. ∎

Example 1.3 (USAMO 1976 P5)**.** If $P(x)$, $Q(x)$, $R(x)$, and $S(x)$ are all polynomials such that

$$P(x^5) + xQ(x^5) + x^2R(x^5) = (x^4 + x^3 + x^2 + x + 1)S(x),$$

prove that $x - 1$ is a factor of $P(x)$.

Solution 3. Denote the 5-th root of unity $\cos \frac{2\pi}{5} + i \sin \frac{2\pi}{5}$ by ζ. Substituting $x = \zeta, \zeta^2, \zeta^3$, we obtain

$$P(1) + \zeta Q(1) + \zeta^2 R(1) = 0,$$
$$P(1) + \zeta^2 Q(1) + \zeta^4 R(1) = 0,$$
$$P(1) + \zeta^3 Q(1) + \zeta^6 R(1) = 0.$$

Eliminating $R(1)$ from the first two equations yields

$$(1 - \zeta^2)P(1) + \zeta^2(1 - \zeta)Q(1) = 0,$$

and eliminating $R(1)$ from the last two equations yields

$$(1 - \zeta^2)P(1) + \zeta^3(1 - \zeta)Q(1) = 0.$$

Eliminating $Q(1)$ from the above two equations, we obtain $(1 - \zeta)P(1) = 0$, which gives $P(1) = 0$. This shows that $x - 1$ is a factor of $P(x)$. ∎

Example 1.4 (Leningrad Math Olympiad 1991)**.** A finite sequence $a_1, a_2, \ldots, a_n$ is called p-balanced if any sum of the form

$$a_k + a_{k+p} + a_{k+2p} + \ldots$$

is the same for any $k = 1, 2, 3, \ldots$. For instance the sequence

$$a_1 = 1, a_2 = 2, a_3 = 3, a_4 = 4, a_5 = 3, a_6 = 2$$

is 3-balanced. Prove that if a sequence with 50 members is p-balanced for $p = 3, 5, 7, 11, 13, 17$, then all its members are equal zero.

Summary — Consider the polynomial $\sum_{i=1}^{n} a_i x^n$.

Solution 4. Let $a_1, a_2, \ldots, a_{50}$ be a sequence. Assume that it is p-balanced for $p \in \{3, 5, 7, 11, 13, 17\}$. For an integer $n \geq 1$, denote the root of unity $\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$ by ζ_n. Let $P(x)$ denote the polynomial $\sum_{i=1}^{n} a_i x^i$.

Let $3 \leq p \leq 17$ be a prime. Since $a_1, a_2, \ldots, a_{50}$ is p-balanced, for any $1 \leq \ell < p$, we obtain

$$P(\zeta_p^\ell) = \sum_{k=1}^{p} (a_k + a_{k+p} + \ldots)\zeta_p^{k\ell}$$

$$= (a_1 + a_{1+p} + \ldots) \sum_{k=1}^{p} \zeta_p^{k\ell}$$

$$= 0,$$

where the final equality follows since $\zeta_p^\ell \neq 1$. This shows that the polynomial $P(x)$ vanishes at the elements of the set

$$\cup_{p\in\{3,5,7,11,13,17\}}\{\zeta_p^\ell \,|\, 1 \leq \ell < p\},$$

which contains

$$\sum_{p\in\{3,5,7,11,13,17\}} (p-1) = 2 + 4 + 6 + 10 + 12 + 16 = 50$$

elements. Moreover, $P(x)$ also vanishes at 0. Note that $P(x)$ is a polynomial of degree 50, and it has at least 51 zeroes. This gives that $P(x) = 0$, and hence, the terms of the sequence $a_1, a_2, \ldots, a_{50}$ are all equal to zero. ∎

Example 1.5. Show that any function $f : \mathbb{R} \to \mathbb{R}$ can be written as a sum of an even function and an odd function.

Solution 5. Consider the functions $f_1, f_{-1} : \mathbb{R} \to \mathbb{R}$ defined by

$$f_\varepsilon(x) = \frac{1}{2}(f(x) + \varepsilon f(\varepsilon x)), \quad \text{for all } x \in \mathbb{R},$$

where $\varepsilon \in \{1, -1\}$. Note that f_1 is an even function, f_{-1} is an odd function, and that $f = f_1 + f_{-1}$ holds. ∎

Remark. It seems from the above solution that we "knew" the functions f_1, f_{-1} beforehand! Here is another solution to the above problem.

Let us assume that f can be expressed as the sum of an even function g and an odd function h. Note that

$$f(x) = g(x) + h(x),$$
$$f(-x) = g(-x) + h(-x)$$

$$= g(x) - h(x)$$

hold for all $x \in \mathbb{R}$. Combining the above, we obtain

$$g(x) = \frac{1}{2}(f(x) + f(-x)),$$

$$h(x) = \frac{1}{2}(f(x) - f(-x))$$

for all $x \in \mathbb{R}$.

Note that this is **NOT a proof**. We have observed that

$$g(x) = \frac{1}{2}(f(x) + f(-x)),$$

$$h(x) = \frac{1}{2}(f(x) - f(-x))$$

holds **relying on the assumption that** g is an even function, and h is an odd function satisfying $f = g + h$. This observation is **made under the hypothesis** that $f = g + h$ where g (resp. h) is even (resp. odd), and **cannot be a proof of the hypothesis**! However, this observation can be used to arrive at the above proof, by defining the functions f_1, f_{-1} as above, and showing that they have the required properties.

Example 1.6. Let ω denote the root of unity $\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$. Show that any function $f : \mathbb{C} \to \mathbb{C}$ can be written as a sum of three functions $f_0, f_1, f_2 : \mathbb{C} \to \mathbb{C}$ satisfying

$$f_i(\omega z) = \overline{\omega}^i z \quad \text{for all } z \in \mathbb{C}.$$

Walkthrough —— Assume that the function f can be expressed as stated, and then determine some of the properties of the functions f_0, f_1, f_2, as done in the above remark. What remains left to do?

Example 1.7. Let $P(x)$ be a monic polynomial with integer coefficients such that all its zeroes lie on the unit circle. Show that all the zeroes of $P(x)$ are roots of unity, that is, $P(x)$ divides $(x^n - 1)^k$ for some positive integers n, k.

The author has learnt the following argument from Mainak Ghosh, during the INMO Training Camp 2025, held at IISER Bhopal.

Walkthrough ——

(a) Use the fundamental theorem of symmetric polynomials, to prove the following claim.

> **Claim** —— Let $f(x)$ be a monic polynomial of degree n with integer coefficients. Let $\alpha_1, \ldots, \alpha_n$ denote its roots, counting multiplicities. Then for any integer $k \geq 1$, there is a monic polynomial of degree n with integer coefficients, having $\alpha_1^k, \alpha_2^k, \ldots, \alpha_n^k$ as its roots.

(b) Applying the Claim, for each integer $k \geq 1$, obtain a monic polynomial $P_k(x)$ with integer coefficients, having degree same as that of $P(x)$, whose roots, counted with multiplicities, are the k-th powers of the roots of $P(x)$.

(c) Note that the polynomials $P_1(x), P_2(x), \dots$ are of the same degree, and the absolute values of the coefficients of any of them are bounded from the above by suitable binomial coefficients, which is smaller than 2^n, where n denotes the degree of $P(x)$. Since these polynomials have integer coefficients, by the pigeonhole principle, it follows that there is a positive integer k_0 such that $P_k(x) = P_{k_0}(x)$ holds for infinitely many positive integers k.

(d) Enumerate the roots of $P(x)$, and matching the roots of $P_k(x)$ with those of $P_{k_0}(x)$ (for suitable k's), we obtain a permutation of n letters. By the pigeonhole principle, infinitely many k's yield the same permutation, which implies that there are positive integers $k \neq \ell$ such that for any root of $P(x)$, its k-th and the ℓ-th powers are equal.

References

[**Che25**]   EVAN CHEN. *The OTIS Excerpts.* Available at https://web.evanchen.cc/excerpts.html. 2025, pp. vi+289 (cited p. 1)