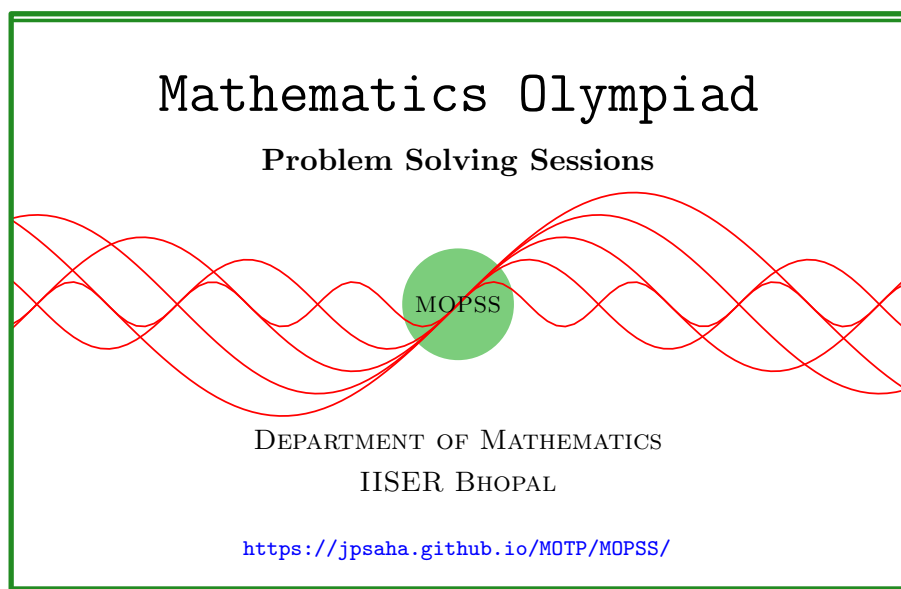


Size of the roots

MOPSS

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Suggested readings

- Evan Chen's advice [On reading solutions](https://blog.evanchen.cc/2017/03/06/on-reading-solutions/), available at <https://blog.evanchen.cc/2017/03/06/on-reading-solutions/>.
- Evan Chen's [Advice for writing proofs/Remarks on English](https://web.evanchen.cc/handouts/english/english.pdf), available at <https://web.evanchen.cc/handouts/english/english.pdf>.
- [Notes on proofs](#) by Evan Chen from [OTIS Excerpts](#) [[Che25](#), Chapter 1].
- [Tips for writing up solutions](https://www.math.utoronto.ca/barbeau/writingup.pdf) by Edward Barbeau, available at <https://www.math.utoronto.ca/barbeau/writingup.pdf>.
- Evan Chen discusses why [math olympiads are a valuable experience for high schoolers](#) in the post on [Lessons from math olympiads](#), available at <https://blog.evanchen.cc/2018/01/05/lessons-from-math-olympiads/>.

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§1 Size of the roots

Some of the following problems have been borrowed from [Goy21].

Example 1.1. Let $f(x)$ and $g(x)$ be nonconstant polynomials with real coefficients such that $f(x^2 + x + 1) = f(x)g(x)$. Show that $f(x)$ has even degree.

Solution 1. For any real root α of $f(x)$, note that $\alpha^2 + \alpha + 1$ is also a real root of $f(x)$ and $\alpha^2 + \alpha + 1 > \alpha$. Hence, if $f(x)$ possesses a real root, it would have real roots of arbitrarily large absolute value. Since $f(x)$ is nonconstant, it follows that $f(x)$ has no real roots. Since $f(x)$ is a polynomial with real coefficients, it follows that $f(x)$ is of even degree. ■

Example 1.2. Find all polynomials P (with complex coefficients) satisfying

$$P(x)P(x+2) = P(x^2).$$

Summary — Note that if α is a root of P , then so are α^2 and $(\alpha - 2)^2$. Considering absolute values, show that P cannot have a root other than 1. Conclude that $P(x) = c(x - 1)^n$.

Solution 2. Let P be a polynomial with complex coefficients satisfying the given condition.

Claim — If P has a root $\alpha \neq 1$, then P has a root which has absolute value larger than the absolute value of α .

Proof of the Claim. Note that if $|\alpha| \leq 1$, then

$$\begin{aligned} |(\alpha - 2)^2| &\geq (2 - |\alpha|)^2 \\ &\geq 1 \\ &\geq |\alpha|. \end{aligned}$$

Moreover, if $|(\alpha - 2)^2| = |\alpha|$ holds, then $|\alpha| = 1$ and $\alpha = 2r$ for some real number $r \geq 0$, and hence, $\alpha = 1$. This shows that if $\alpha \neq 1$, and $|\alpha| \leq 1$, then $|(\alpha - 2)^2| > |\alpha|$ holds. Also note that if $|\alpha| > 1$, then $|\alpha^2| > |\alpha|$ holds. Since $\alpha^2, (\alpha - 2)^2$ are also roots of P , the Claim follows. \square

If P has a root other than 1, then applying the above Claim to a root of P having the largest absolute value yields a contradiction. Hence, P is a constant polynomial, or the only root of P is equal to 1.

If P is a constant, then P is equal to 0 or 1. If P is a nonconstant polynomial, then P is equal to $c(x - 1)^n$ for some $c \in \mathbb{C} \setminus \{0\}$, and an integer $n \geq 1$. Using the hypothesis, it follows that $c = 1$.

Observing that the polynomials $0, 1, (x - 1)^n$ satisfy the given condition, we conclude that there are all the required polynomials. \blacksquare

Example 1.3. Does there exist a polynomial $f(x)$ satisfying

$$xf(x - 1) = (x + 1)f(x)?$$

Problem 4.12 of [Putnam training problems](#) by Miguel A. Lerma.

Solution 3. Note that the zero polynomial is the only constant polynomial which satisfies the given condition.

Suppose there exists a nonconstant polynomial $f(x)$ satisfying the given condition.

Claim — Any root of $f(x)$ is an integer.

Proof of the Claim. On the contrary, let us assume that α is root of $f(x)$, and that α is not an integer. Note that for any root β of $f(x)$ in $\mathbb{C} \setminus \mathbb{Z}$, the element $\beta - 1$ is also a root of $f(x)$ lying in $\mathbb{C} \setminus \mathbb{Z}$. It follows that for any integer $k \geq 1$, the element $\alpha - k$ is also a root of $f(x)$. Taking k to be an integer larger than $|\alpha| + \sum_{\gamma \in \mathbb{C}, f(\gamma)=0} |\gamma|$, we obtain

$$|\alpha - k| \geq k - |\alpha| > |\gamma|,$$

and hence $f(x)$ has a root having absolute value larger than that of any of its roots, which is impossible. \square

A similar argument can be used to prove that $f(x)$ does not vanish at any negative integer. Moreover, noting that if $f(x)$ vanishes at an integer $n \geq -1$, then $f(x)$ also vanishes at $n + 1 \geq -1$, it can be proved using a similar argument that the roots of $f(x)$ are ≤ -2 . This contradicts the assumption that there exists a nonconstant polynomial satisfying the given condition. Consequently, the zero polynomial is the only polynomial that satisfies the given condition. \blacksquare

Example 1.4. Find all polynomials $P(x)$ satisfying

$$xP(x - 1) = (x - 20)P(x).$$

Problem 2 of the [Problem session](#) for October 28, Fall 2020, Putnam Club.

Solution 4. Note that the zero polynomial is the only constant polynomial which satisfies the given condition.

Suppose $P(x)$ is a nonconstant polynomial satisfying the given condition.

Claim — Any root of $P(x)$ is an integer.

Proof of the Claim. On the contrary, let us assume that α is root of $P(x)$, and that α is not an integer. Note that for any root β of $P(x)$ in $\mathbb{C} \setminus \mathbb{Z}$, the element $\beta - 1$ is also a root of $P(x)$ lying in $\mathbb{C} \setminus \mathbb{Z}$. It follows that for any integer $k \geq 1$, the element $\alpha - k$ is also a root of $P(x)$. Taking k to be an integer larger than $|\alpha| + \sum_{\gamma \in \mathbb{C}, P(\gamma)=0} |\gamma|$, we obtain

$$|\alpha - k| \geq k - |\alpha| > |\gamma|,$$

and hence $P(x)$ has a root having absolute value larger than that of any of its roots, which is impossible. \square

A similar argument can be used to prove that no root of $P(x)$ is ≤ -1 or ≥ 20 . In other words, the roots of $P(x)$ are among $0, 1, 2, \dots, 19$. The given condition implies that if $0 \leq i \leq 19$ is a root of $P(x)$, then all of the integers $0, 1, 2, \dots, 19$ are roots of $P(x)$. Hence, $P(x)$ is equal to $cx^{n_0}(x-1)^{n_1} \dots (x-19)^{n_{19}}$ for some $c \in \mathbb{C}$, and positive integers n_0, n_1, \dots, n_{19} . Using the condition $xP(x-1) = (x-20)P(x)$, it follows that the integers n_0, \dots, n_{19} are equal to 1. This shows that $P(x)$ is equal to $cx(x-1)(x-2) \dots (x-20)$. Moreover, any polynomial of this form satisfies the given condition. Hence, the polynomials satisfying the given condition are precisely of the above form. \blacksquare

Example 1.5 (India INMO 2018 P4). Find all polynomials $P(x)$ with real coefficients such that $P(x^2 + x + 1)$ divides $P(x^3 - 1)$.

Walkthrough —

- (a) Show that if α is a root of $P(x)$, then $P(x)$ vanishes at $(\beta_1 - 1)\alpha$ and $(\beta_2 - 1)\alpha$, where β_1, β_2 are the roots of $x^2 + x + 1 = 0$.
- (b) If α is nonzero, then show that one of $(\beta_1 - 1)\alpha$ and $(\beta_2 - 1)\alpha$ is larger than α in absolute value.

Solution 5. Let us establish the following claim.

Claim — Let α denote a nonzero root of $P(x)$ in \mathbb{C} . Then for some $\beta \in \mathbb{C}$ satisfying $\beta^2 + \beta + 1 = 0$, the element $(\beta - 1)\alpha$ is a root of $P(x)$

and is larger than α in absolute value.

Proof of the Claim. Let β_1, β_2 denote the roots of $x^2 + x + 1 = \alpha$ in \mathbb{C} . Since $P(x^2 + x + 1)$ divides the polynomial $P(x^3 - 1)$, it follows that for any $1 \leq i \leq 2$,

$$\beta_i^3 - 1 = (\beta_i - 1)(\beta_i^2 + \beta_i + 1) = (\beta_i - 1)\alpha$$

is a root of $P(x)$. Noting that

$$\begin{aligned} |\beta_1 - 1| + |\beta_2 - 1| &\geq |\beta_1 + \beta_2 - 2| \\ &= |-1 - 2| \\ &> 2, \end{aligned}$$

we obtain that at least one of $\beta_1 - 1, \beta_2 - 1$ has absolute value larger than 1. Consequently, $|(\beta_i - 1)\alpha| > |\alpha|$ holds for some $1 \leq i \leq 2$. This proves the Claim. \square

If $P(x)$ has a nonzero root in \mathbb{C} , then applying the above Claim to a root of $P(x)$ of largest absolute value, we would obtain a contradiction. Hence, $P(x)$ is equal to cx^n for some $c \in \mathbb{R}$ and an integer $n \geq 0$. Moreover, any polynomial of this form satisfies the given condition. Consequently, these are all the polynomials satisfying the given condition. \blacksquare

Remark. Note that proving $|(\beta_1 - 1)(\beta_2 - 1)| > 1$, in order to conclude that at least one of $\beta_1 - 1, \beta_2 - 1$ has absolute value larger than 1, does not seem to work. Moreover, $|(\beta_1 - 1)(\beta_2 - 1)|$ is smaller than $\frac{1}{2}(|\beta_1 - 1| + |\beta_2 - 1|)$. **Unsurprisingly**, a lower bound for the bigger quantity can easily be obtained.

Example 1.6 (IMOSL 1979 Bulgaria). Find all polynomials $f(x)$ with real coefficients satisfying

$$f(x)f(2x^2) = f(2x^3 + x).$$

Solution 6. Note that $0, 1$ are the only constant polynomials satisfying the given condition.

Let $f(x)$ be a nonconstant polynomial with real coefficients satisfying the given condition.

Claim — The roots of $f(x)$ in \mathbb{C} are of absolute value at most 1.

Proof of the Claim. If α is a root of $f(x)$ in \mathbb{C} , then $2\alpha^3 + \alpha$ is a root of $f(x)$. If $|\alpha| > 1$, then

$$|2\alpha^3 + \alpha| \geq 2|\alpha|^3 - |\alpha| > |\alpha|$$

holds. If some root of $f(x)$ has absolute value larger than 1, then taking α to be a root of $f(x)$ with largest absolute value, we would obtain a contradiction. This proves the Claim. \square

Claim — The equality $f(0) = 1$ holds.

Proof of the Claim. Substituting $x = 0$, it follows that $f(0)^2 = f(0)$, and hence $f(0) = 0$ or $f(0) = 1$.

On the contrary, suppose $f(0) = 0$ holds. Write $f(x) = x^k g(x)$ where k is a positive integer, and $g(x)$ is a polynomial with real coefficients with $g(0) \neq 0$. The given condition translates to

$$x^k (2x^2)^k g(x) g(2x^2) = (2x^3 + x)^k g(2x^3 + x),$$

which yields

$$(2x^2)^k g(x) g(2x^2) = (2x^2 + 1)^k g(2x^3 + x).$$

Substituting $x = 0$, we obtain $g(0) = 0$, which is impossible. This proves the Claim. \square

By the above Claim, the product of the absolute values of the roots of $f(x)$ is equal to 1. By the first Claim, these absolute values are at most 1. It follows that the roots of $f(x)$ are of absolute value 1.

Let α be a root of $f(x)$. Note that $2\alpha^3 + \alpha$ is also a root of $f(x)$, and we have that

$$|\alpha| = |2\alpha^3 + \alpha| = 1,$$

which yields $|2\alpha^2 + 1| = |\alpha| = 1$. This gives

$$(2\alpha^2 + 1)(2\bar{\alpha}^2 + 1) = 1.$$

Combining it with $|\alpha| = 1$, we obtain

$$\alpha^2 + \bar{\alpha}^2 = -2.$$

Since $|\alpha| = 1$, it follows that $\alpha^2 = -1$, and hence, $\alpha = i$ or $\alpha = -i$. So, the polynomial $f(x)$ is equal to $c(x+i)^a(x-i)^b$ for some $c \in \mathbb{C} \setminus \{0\}$, and some nonnegative integers a, b . Since $f(x)$ has real coefficients, it follows that $a = b$, and c lies in \mathbb{R} . This gives that $f(x) = c(x^2 + 1)^a$. Using $f(0) = 1$, we obtain $c = 1$, and hence $f(x) = (x^2 + 1)^a$.

Note that if $g(x)$ denotes the polynomial $(x^2 + 1)^k$, where k is a positive integer, then

$$\begin{aligned} g(x)g(2x^2) &= ((x^2 + 1)((2x^2)^2 + 1))^k \\ &= ((4x^6 + 4x^4 + x^2 + 1))^k \\ &= (((2x^3 + x)^2 + 1))^k \\ &= g(2x^3 + x). \end{aligned}$$

We conclude that the polynomials satisfying the given condition are precisely the constant polynomial 0, 1, and the polynomials of the form $(x^2 + 1)^k$ for some positive integer k . \blacksquare

References

- [Che25] EVAN CHEN. *The OTIS Excerpts*. Available at <https://web.evanchen.cc/excerpts.html>. 2025, pp. vi+289
- [Goy21] ROHAN GOYAL. “Polynomials”. Available at <https://www.dropbox.com/s/yo31nat6z5ggaue/Polynomials.pdf?dl=0>. 2021 (cited p. 2)