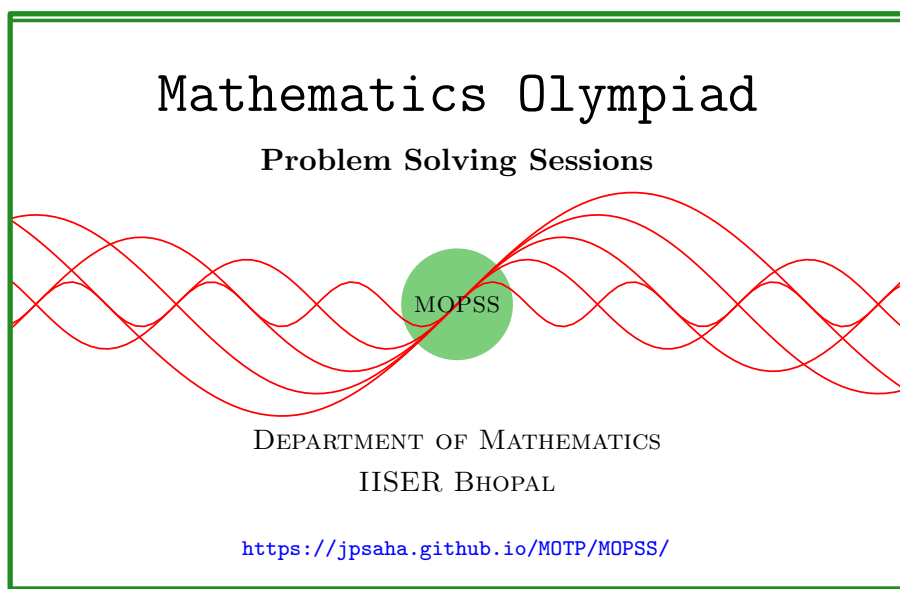


Rational and irrational numbers

MOPSS

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Suggested readings

- Evan Chen's advice [On reading solutions](https://blog.evanchen.cc/2017/03/06/on-reading-solutions/), available at <https://blog.evanchen.cc/2017/03/06/on-reading-solutions/>.
- Evan Chen's [Advice for writing proofs/Remarks on English](https://web.evanchen.cc/handouts/english/english.pdf), available at <https://web.evanchen.cc/handouts/english/english.pdf>.
- [Notes on proofs](#) by Evan Chen from [OTIS Excerpts](#) [[Che25](#), Chapter 1].
- [Tips for writing up solutions](https://www.math.utoronto.ca/barbeau/writingup.pdf) by Edward Barbeau, available at <https://www.math.utoronto.ca/barbeau/writingup.pdf>.
- Evan Chen discusses why [math olympiads are a valuable experience for high schoolers](#) in the post on [Lessons from math olympiads](#), available at <https://blog.evanchen.cc/2018/01/05/lessons-from-math-olympiads/>.

List of problems and examples

1.1	Example (Putnam 2009 B1)	2
1.2	Example	2
1.3	Example (Moscow Math Circles)	2
1.4	Example (India RMO 2013d P4)	3
1.5	Example (All-Russian MO 2001–2002 Final stage Grade 11 P1)	3
1.6	Example (British Mathematical Olympiad Round 1 2004/5 P5)	4
1.7	Example (Junior Balkan MO TST 1999)	5
1.8	Example (India RMO 2019a P1)	7

§1 Rational and irrational numbers

Example 1.1 (Putnam 2009 B1). Show that every positive rational number can be written as a quotient of products of factorials of (not necessarily distinct) primes. For example,

$$\frac{10}{9} = \frac{2! \cdot 5!}{3! \cdot 3! \cdot 3!}.$$

Example 1.2. Let $n \geq 2$ be an integer. Let m be the largest integer which is less than or equal to n , and which is a power of 2. Put $\ell_n =$ the least common multiple of $1, 2, \dots, n$. Show that ℓ_n/m is odd, and that for every integer $k \leq n, k \neq m, \ell_n/k$ is even. Hence prove that

$$1 + \frac{1}{2} + \dots + \frac{1}{n}$$

is not an integer.

Example 1.3 (Moscow Math Circles). Does there exist irrational numbers x, y with $x > 0$ such that x^y is rational?

Summary — Consider $\sqrt{2}^{\sqrt{2}}$.

Walkthrough —

- (a) Consider $\sqrt{2}^{\sqrt{2}}$.
- (b) If $\sqrt{2}^{\sqrt{2}}$ is rational, then we are done by taking $x = y = \sqrt{2}$.
- (c) If $\sqrt{2}^{\sqrt{2}}$ is irrational, then can you find out suitable x, y ?

Solution 1. Consider $\sqrt{2}^{\sqrt{2}}$. If $\sqrt{2}^{\sqrt{2}}$ is rational, then we may take $x = y = \sqrt{2}$. If $\sqrt{2}^{\sqrt{2}}$ is irrational, then taking $x = \sqrt{2}^{\sqrt{2}}$ and $y = \sqrt{2}$, we find that

$$x^y = \left(\sqrt{2}^{\sqrt{2}} \right)^{\sqrt{2}} = \left(\sqrt{2} \right)^2 = 2,$$

which is a rational number. ■

Example 1.4 (India RMO 2013d P4). Let x be a nonzero real number such that $x^4 + \frac{1}{x^4}$ and $x^5 + \frac{1}{x^5}$ are both rational numbers. Prove that $x + \frac{1}{x}$ is a rational number.

Summary — Consider the difference

$$\left(x^5 + \frac{1}{x^5}\right) - \left(x^4 + \frac{1}{x^4}\right)\left(x + \frac{1}{x}\right).$$

Solution 2. For a positive integer n , put $y_n = x^n + \frac{1}{x^n}$. Note that

$$y_5 = y_1 y_4 - y_3 = y_1 y_4 - y_1(y_2 - 1) = y_1(y_4 - y_2 + 1),$$

and $y_4 - y_2 + 1$ is nonzero, otherwise, we would get $y_2^2 - y_2 - 1 = 0$, that is, $y_2 = \frac{1 \pm \sqrt{5}}{2}$, which shows that $y_4 = \frac{3 \pm \sqrt{5}}{2}$, which contradicts that y_4 is rational. To show that y_1 is rational, it suffices to show that y_2 is rational. Observe that

$$y_{10} = y_2 y_8 - y_6 = y_2 y_8 - y_2 y_4 + y_2 = y_2(y_8 - y_4 + 1),$$

and $y_8 - y_4 + 1 \neq 0$ (otherwise, we would obtain $y_4^2 - y_4 - 1 = 0$, which contradicts the rationality of y_4). Since y_{10}, y_8, y_4 are rational, it follows that y_2 is rational. Using

$$y_5 = y_1(y_4 - y_2 + 1),$$

and that $y_4 - y_2 + 1$ is nonzero, we conclude that y_1 is rational. ■

Example 1.5 (All-Russian MO 2001–2002 Final stage Grade 11 P1). Real numbers x and y are such that $x^p + y^q$ is rational for any different odd primes p, q . Show that x and y are rational.

Solution 3. The given condition implies that for any three distinct primes p, q, r ,

$$x^p - x^q, x^q - x^r$$

are rational since

$$x^p - x^q = (x^p + y^r) - (x^q + y^r), x^q - x^r = (x^q + y^p) - (x^r + y^p).$$

It follows that

$$a = x^7 - x^5, b = x^5 - x^3$$

are rational. If $b = 0$, then $x = 0$ or $x = \pm 1$, and hence x is rational. If $b \neq 0$, then note that $x^2 = \frac{a}{b}$, which shows that x^2 is rational. Observing that

$$b = x^2(x^2 - 1)x,$$

it follows that if $b \neq 0$, then x is rational. The rationality of y follows similarly. ■

Example 1.6 (British Mathematical Olympiad Round 1 2004/5 P5). Let S be a set of rational numbers with the following properties:

1. $\frac{1}{2} \in S$,
2. If $x \in S$, then both $\frac{1}{x+1} \in S$ and $\frac{x}{x+1} \in S$.

Prove that S contains all rational numbers in the interval $0 < x < 1$.

Walkthrough —

(a) Since $\frac{1}{2}$ lies in S , by the second condition, it follows that $\frac{2}{3}$ lies in S and so does $\frac{1}{3}$.

(b) Taking $x = \frac{1}{3}$, it follows that

$$\frac{3}{4}, \frac{1}{4}$$

lie in S . **Note that we have showed that S contains all the rationals between 0 and 1 with denominator at most 4.**

(c) Taking $x = \frac{2}{3}$, it follows that

$$\frac{2}{5}, \frac{3}{5}$$

lie in S . We are **not in a position** to conclude that S contains all the rationals between 0 and 1 with denominator at most 5.

(d) Taking $x = \frac{1}{4}$, it follows that

$$\frac{1}{5}, \frac{4}{5}$$

lie in S . It follows that S contains all the rationals between 0 and 1 with denominator at most 5.

(e) Does the above provide any insight to conclude that S contains all the rationals between 0 and 1? For instance, can one expect the following (and then prove, or realize that it is false, or argue along different lines)?

For a rational number x lying in S , the rationals

$$\frac{1}{x+1}, \frac{x}{x+1}$$

have denominators larger^a than that of x .

^aOften, while being naive, one takes the liberty to write **larger** to mean **no smaller**, that is, **greater than or equal to**. But this is **NOT allowed** while writing down a solution.

Or, stated in a different way,

A rational number lying in $(0, 1)$ can be obtained from a rational number lying in $(0, 1)$ with smaller denominator by applying one of the maps

$$x \mapsto \frac{1}{x+1}, x \mapsto \frac{x}{x+1}.$$

Solution 4. It suffices to establish the following.

Claim — For any integer $k \geq 2$, all the rationals lying in $(0, 1)$ with denominators not exceeding k lie in S , that is, we have

$$\left\{ \frac{1}{\ell}, \frac{2}{\ell}, \dots, \frac{\ell-2}{\ell}, \frac{\ell-1}{\ell} \right\} \subseteq S \quad \text{for all } 2 \leq \ell \leq k. \quad (1)$$

Proof of the Claim. Eq. (1) holds for $k = 2$ from condition (1). Suppose Eq. (1) holds for $k = n - 1$ for some integer $n \geq 3$. Let m be an integer satisfying $1 \leq m < n$. Using the induction hypothesis, we will show that $\frac{m}{n}$ lies in S . Note that for $0 < x < 1$, the inequalities

$$0 < \frac{x}{x+1} < \frac{1}{2}, \frac{1}{2} < \frac{1}{x+1} < 1$$

hold. Using Condition (1), it follows that $\frac{m}{n}$ lies in S if $\frac{m}{n} = \frac{1}{2}$. If $0 < \frac{m}{n} < \frac{1}{2}$, then

$$\frac{x}{x+1} = \frac{m}{n}$$

holds for $x = \frac{m}{n-m}$, which is a rational number lying in $(0, 1)$ with denominator $\leq n - 1$, and by induction hypothesis, the set S contains $\frac{m}{n}$. Moreover, if $\frac{1}{2} < \frac{m}{n} < 1$, then

$$\frac{1}{x+1} = \frac{m}{n}$$

holds for $x = \frac{n-m}{m}$, which is a rational number lying in $(0, 1)$ with denominator $\leq n - 1$, and by induction hypothesis, the set S contains $\frac{m}{n}$. We conclude that for any integer $n \geq 3$, Eq. (1) holds for $k = n$ if it holds for $k = n - 1$. \square

■

Example 1.7 (Junior Balkan MO TST 1999). Let S be a set of rational numbers with the following properties:

1. $\frac{1}{2} \in S$,
2. If $x \in S$, then both $\frac{x}{2} \in S$ and $\frac{1}{x+1} \in S$.

Prove that S contains all the rational numbers from the interval $(0, 1)$.

Walkthrough —

- (a) Taking $x = \frac{1}{2}$, it follows that S contains $\frac{2}{3}$, and hence it also contains $\frac{1}{3}$.
- (b) Taking $x = \frac{1}{2}$, it follows that S contains $\frac{1}{4}$. Next, taking $x = \frac{1}{3}$, we obtain that S contains $\frac{3}{4}$.
- (c) Applying the map $x \mapsto \frac{1}{x+1}$ to $x = \frac{2}{3}, \frac{1}{4}$, it follows that S contains $\frac{3}{5}, \frac{4}{5}$. Since S contains $\frac{4}{5}$, the set S also contains $\frac{2}{5}, \frac{1}{5}$.
- (d) Applying $x \mapsto \frac{1}{x+1}$ to $x = \frac{1}{5}$, it follows that S contains $\frac{5}{6}$. Note that S contains $\frac{4}{6} = \frac{2}{3}, \frac{3}{6} = \frac{1}{2}, \frac{2}{6} = \frac{1}{3}$. It also follows that S contains $\frac{1}{6}$.
- (e) Does the above provide any insight into the problem? Can one expect the following?

The rationals lying in $(\frac{1}{2}, 1)$ can be obtained by applying the map $x \mapsto \frac{1}{x+1}$ to the rationals lying in $(0, 1)$ with small denominators. Moreover, a rational number r lying in $(0, \frac{1}{2})$ can be obtained by applying the map $x \mapsto \frac{x}{2}$ to the rationals lying in $(\frac{1}{2}, 1)$, more specifically, to those rationals with denominators at most the denominator of r .

Solution 5. It suffices to establish the following.

Claim — For any integer $k \geq 2$, all the rationals lying in $(0, 1)$ with denominators not exceeding k lie in S , that is, we have

$$\left\{ \frac{1}{\ell}, \frac{2}{\ell}, \dots, \frac{\ell-2}{\ell}, \frac{\ell-1}{\ell} \right\} \subseteq S \quad \text{for all } 2 \leq \ell \leq k. \quad (2)$$

Proof of the Claim. Note that Eq. (2) holds for $k = 2$ by hypothesis. Suppose Eq. (2) holds for $k = n - 1$ where $n \geq 3$ is an integer. Let m be an integer satisfying $1 \leq m < n$. Using the induction hypothesis, we will show that $\frac{m}{n}$ lies in S . Note that for $0 < x < 1$, the inequalities

$$\frac{1}{2} < \frac{x}{x+1} < 1$$

hold. If $\frac{m}{n}$ is equal to $\frac{1}{2}$, then it lies in S by hypothesis. If $\frac{m}{n}$ lies in $(\frac{1}{2}, 1)$, then

$$\frac{1}{x+1} = \frac{m}{n}$$

holds for $x = \frac{n-m}{m}$, which is a rational number lying in $(0, 1)$ with denominator $\leq n - 1$, and by induction hypothesis, the set S contains $\frac{m}{n}$. If $\frac{m}{n}$ lies in $(0, \frac{1}{2})$, then the rational number $\frac{2m}{n}$ lies in $(\frac{1}{2}, 1)$, and if it has denominator $< n$ (when expressed in its least form), then it is an element of S by the induction hypothesis, and if it has denominator equal to n (when expressed in its least

form), then by the above argument, it lies in S , and consequently, S contains $\frac{m}{n}$. We conclude that for any integer $n \geq 3$, Eq. (2) holds for $k = n$ if it holds for $k = n - 1$. \square

■

Example 1.8 (India RMO 2019a P1). Suppose x is a nonzero real number such that both x^5 and $20x + \frac{19}{x}$ are rational numbers. Prove that x is a rational number.

Solution 6. Let us establish the following more general claim.

Claim — Let x be a nonzero real number and r be a rational number such that x^5 and $x + \frac{r}{x}$ are rational numbers. Then x is a rational number.

Proof of the Claim. Note that

$$x^2 + \frac{r^2}{x^2}, x^3 + \frac{r^3}{x^3}$$

are rational. Write $s = x^5$ and note that

$$x^3 + \frac{r^3}{x^3} = \frac{r^3}{s}x^2 + \frac{s}{x^2}.$$

It follows that there are rational numbers a, b such that

$$\begin{pmatrix} 1 & r^2 \\ \frac{r^3}{s} & s \end{pmatrix} \begin{pmatrix} x^2 \\ \frac{1}{x^2} \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

holds. If the determinant of $\begin{pmatrix} 1 & r^2 \\ \frac{r^3}{s} & s \end{pmatrix}$ is nonzero, then it follows that x^2 is rational, and using the rationality of x^5 , we obtain that x is rational. If the determinant of $\begin{pmatrix} 1 & r^2 \\ \frac{r^3}{s} & s \end{pmatrix}$ is zero, then we get $s^2 = r^5$, which implies that $x^{10} = r^5$, and hence, x^2 is rational, and consequently, so is x . \square

■

References

[Che25] EVAN CHEN. *The OTIS Excerpts*. Available at <https://web.evanchen.cc/excerpts.html>. 2025, pp. vi+289