Quadratic polynomials

MOPSS

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Mathematics Olympiad

Problem Solving Sessions

MOPSS

Department of Mathematics
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https://jpsaha.github.io/MOTP/MOPSS/

Suggested readings

- Evan Chen's advice On reading solutions, available at https://blog.evanchen.cc/2017/03/06/on-reading-solutions/.
- Evan Chen's Advice for writing proofs/Remarks on English, available at https://web.evanchen.cc/handouts/english/english.pdf.
- Notes on proofs by Evan Chen from OTIS Excerpts [**Che25**, Chapter 1].
- Tips for writing up solutions by Edward Barbeau, available at https://www.math.utoronto.ca/barbeau/writingup.pdf.
- Evan Chen discusses why math olympiads are a valuable experience for high schoolers in the post on Lessons from math olympiads, available at https://blog.evanchen.cc/2018/01/05/lessons-from-math-olympiads/.

List of problems and examples

§1 Quadratic polynomials

Example 1.1 (Hungary MO 2001/02, Grades 11 and 12 — technical schools, P2)**.** Consider the following 2000 equations:

$$\begin{gathered}1x^2 + 2 \cdot 2x + 3 = 0\\ 2x^2 + 2 \cdot 3x + 4 = 0\\ 3x^2 + 2 \cdot 4x + 5 = 0\\ \vdots\\ 2000x^2 + 2 \cdot 2001x + 2002 = 0.\end{gathered}$$

For each equation, consider the product of the sum of the real roots and the sum of their reciprocals (if it exists). What is the product of these products?

Solution 1. Note that a quadratic polynomial with nonzero constant term has nonzero roots, and hence the reciprocals of its roots exist. Moreover, if α, β denote the roots of a quadratic polynomial $ax^2 + bx + c$ with $ac \neq 0$, then $1/\alpha, 1/\beta$ are the roots of quadratic polynomial $cx^2 + bx + a$. Note that

$$\alpha + \beta = -\frac{b}{a}, \frac{1}{\alpha} + \frac{1}{\beta} = -\frac{b}{c}$$

Since

$$(2(n+1))^2 - 4n(n+2) = 4n^2 + 8n + 4 - 4n^2 - 8n = 4 \geq 0,$$

it follows that the given 2000 equations have real roots. The product of the products considered is equal to

$$\begin{aligned}
&\frac{2^2\cdot 2^2}{1\cdot 3}\times\frac{2^2\cdot 3^2}{2\cdot 4}\times\frac{2^2\cdot 4^2}{3\cdot 5}\times\frac{2^2\cdot 5^2}{4\cdot 6}\times\cdots\times\frac{2^2\cdot 2000^2}{1999\cdot 2001}\times\frac{2^2\cdot 2001^2}{2000\cdot 2002}\\
&=\frac{4^{2000}}{1\cdot 2}\frac{2^2\cdot 3^2\cdot\dots\cdot 2001^2}{3^2\cdot 4^2\cdot\dots\cdot 2000^2}\frac{1}{2001\cdot 2002}\\
&=\frac{4^{2000}\cdot 2^2\cdot 2001^2}{1\cdot 2\cdot 2001\cdot 2002}\\
&=\frac{2001}{1001}\times 2^{4000}.
\end{aligned}$$

■

Example 1.2 (Canada CMO 1971 P4)**.** Determine all real numbers a such that the two polynomials x^2+ax+1 and x^2+x+a have at least one root in common.

Solution 2. Let a be a real nuber such that the two polynomials x^2+ax+1 and x^2+x+a have at least one root in common. Let α denote a common root of these polynomials. The equations

$$\alpha^2+a\alpha+1=0, \alpha^2+\alpha+a=0$$

yield

$$(a-1)\alpha=a-1.$$

If $a\neq 1$, then $\alpha=1$ and hence $a=-2$. This proves that $a=1,-2$.

If $a=1$, then the given polynomials have at least one root in common. If $a=-2$, then the given polynomials vanish at 1.

We conclude that $a=1,-2$ are precisely all the real numbers such that the given polynomials have at least one common root. ■

Example 1.3 (India RMO 2003 P6)**.** Find all real numbers a for which the equation

$$x^2+(a-2)x+1=3|x|$$

has exactly three distinct real solutions for x.

Solution 3. Let a be a real number such that $x^2+(a-2)x+1=3|x|$ has exactly three distinct real solutions. Note that the equation

$$(x^2+(a-2)x+1-3x)(x^2+(a-2)x+1+3x)=0$$

also has exactly three distinct real solutions. It follows that the discriminant of one of the polynomials $x^2+(a-2)x+1-3x,\ x^2+(a-2)x+1+3x$

vanishes, and the discriminant of the other is positive. The discriminants of these polynomials are

$$\begin{aligned}(a-5)^2-4 &= a^2-10a+21\\ &= (a-3)(a-7),\\ (a+1)^2-4 &= a^2+2a-3\\ &= (a+3)(a-1)\end{aligned}$$

respectively. It follows that a does not belong to $(-\infty,-3)\cup(-3,1)\cup(1,3)\cup(3,7)\cup(7,\infty)$, or equivalently, a belongs to $\{-3,1,3,7\}$.

Let us determine whether the given equation has three distinct real roots if a lies in $\{-3,1,3,7\}$. Let us assume that a lies in $\{-3,1,3,7\}$. Note that then one of $(a-3)(a-7),(a+3)(a-1)$ vanishes and another is positive, and consequently, one of the polynomials

$$x^2+(a-2)x+1-3x, x^2+(a-2)x+1+3x$$

have distinct real roots, and the other has a double root, which is a real number. Observe that if $x^2+(a-2)x+1-3x$ has a double root, then that root is equal to $-\frac{a-5}{2}$. Note that

$$\begin{aligned}\left(\frac{a-5}{2}\right)^2-(a+1)\left(\frac{a-5}{2}\right)+1 &= \frac{1}{4}\left(a^2-10a+25-2a^2+8a+10+4\right)\\ &= \frac{1}{4}\left(-a^2-2a+39\right),\end{aligned}$$

which has a negative discriminant. This shows that if $x^2+(a-2)x+1-3x$ has a real double root, then that cannot be a zero of $x^2+(a-2)x+1+3x$. Also note that

$$\begin{aligned}\left(\frac{a+1}{2}\right)^2-(a-5)\frac{a+1}{2}+1 &= \frac{1}{4}\left(a^2+2a+1-2a^2+8a+10+4\right)\\ &= \frac{1}{4}\left(-a^2+10a+15\right),\end{aligned}$$

whose roots are not integers. Using that a is an integer, it follows that if $x^2+(a-2)x+1+3x$ has a real double root, then that cannot be a root of $x^2+(a-2)x+1-3x$. We conclude that if a lies in $\{-3,1,3,7\}$, then

$$(x^2+(a-2)x+1-3x)(x^2+(a-2)x+1+3x)=0$$

has exactly three distinct real solutions, or equivalently, the equation

$$x^2+(a-2)x+1=3|x|$$

has exactly three distinct real solutions for x.

So the required real numbers are $a=-3,1,3,7$. ■

Example 1.4 (All-Russian MO 2007 Grade 8 P1). If a, b, c are real numbers, show that at least one of the equations

$$
\begin{aligned}
x^2 + (a-b)x + (b-c) &= 0, \\
x^2 + (b-c)x + (c-a) &= 0, \\
x^2 + (c-a)x + (a-b) &= 0
\end{aligned}
$$

has a real solution.

Solution 4. The sum of the discriminants of the above quadratic polynomials is

$$
\begin{aligned}
&(a-b)^2 - 4(b-c) + (b-c)^2 - 4(c-a) + (c-a)^2 - 4(a-b) \\
&= (a-b)^2 + (b-c)^2 + (c-a)^2,
\end{aligned}
$$

which is positive if not all of a, b, c are equal. Consequently, if not all of the three real numbers a, b, c are equal, then at least one of the quadratic polynomials

$$x^2 + (a-b)x + (b-c), x^2 + (b-c)x + (c-a), x^2 + (c-a)x + (a-b)$$

has positive discriminant, and hence admits real solutions. Moreover, if all of a, b, c are equal, at least one (in fact, all) of the above polynomials admits a real root. ■

Example 1.5 (India RMO 2007 P3). Find all pairs (a, b) of real numbers such that whenever α is a root of $x^2 + ax + b = 0$, $\alpha^2 - 2$ is also a root of the equation.

Solution 5. Let a, b be real numbers such that for any root α of $x^2+ax+b = 0$, $\alpha^2 - 2$ is also a root. Denote the roots of $x^2 + ax + b$ by α, β. There are the following possibilities.

(1) $\alpha^2 - 2 = \alpha$, $\beta^2 - 2 = \beta$,

(2) $\alpha^2 - 2 = \beta$, $\beta^2 - 2 = \alpha$,

(3) $\alpha^2 - 2 = \beta^2 - 2 = \alpha$,

(4) $\alpha^2 - 2 = \beta^2 - 2 = \beta$.

If $\alpha = \beta$, then these four cases are equivalent to

$$\alpha^2 - 2 = \beta^2 - 2 = \alpha = \beta,$$

which shows that α is equal to 2 or -1, and hence (a, b) is equal to $(-4, 4)$ or $(2, 1)$.

It remains to consider the case that $\alpha \neq \beta$, which we assume from now on.

In Case (1), α, β satisfy the equation $X^2 - X - 2 = 0$. So (α, β) is equal to $(2, -1)$ or $(-1, 2)$, and hence (a, b) is equal to $(-1, -2)$.

In Case (2), we have $\alpha^2 - \beta^2 = \beta - \alpha$, which gives $\alpha + \beta = -1$ (since $\alpha \neq \beta$). So

$$\alpha\beta = \frac{1}{2}(\alpha+\beta)^2 - \frac{1}{2}(\alpha^2+\beta^2) = \frac{1}{2}(\alpha+\beta)^2 - \frac{1}{2}(\alpha+\beta+4) = \frac{1}{2} - \frac{3}{2} = -1.$$

This shows that α, β are roots of the quadratic polynomial $x^2 + x - 1$, and hence, (a, b) is equal to $(1, -1)$.

In Case (3), note that α is equal to 2 or -1. Using $\beta^2 = 2 + \alpha$ and $\alpha \neq \beta$, it follows that (α, β) is equal to $(2, -2)$ or $(-1, 1)$, and hence (a, b) is equal to $(0, -4)$ or $(0, -1)$.

Similarly, in Case (4), (α, β) is equal to $(-2, 2)$ or $(-1, 1)$, which shows (a, b) is equal to $(0, -4)$ or $(0, -1)$.

So (a, b) is equal to one of $(-4, 4), (2, 1), (-1, -2), (1, -1), (0, -4), (0, -1)$.

Moreover, if (a, b) is equal to any of these six pairs, then it can be checked that for any root α of $x^2 + ax + b = 0$, $\alpha^2 - 2$ is also a root.

We conclude that all the required pairs are $(-4, 4)$, $(2, 1)$, $(-1, -2)$, $(1, -1)$, $(0, -4)$, $(0, -1)$. ■

Example 1.6 (India RMO 2010 P2)**.** Let

$$P_1(x) = ax^2 - bx - c, P_2(x) = bx^2 - cx - a, P_3(x) = cx^2 - ax - b$$

be three quadratic polynomials where a, b, c are nonzero real numbers. Suppose there exists a real number α such that $P_1(\alpha) = P_2(\alpha) = P_3(\alpha)$. Prove that $a = b = c$.

Solution 6. Since $P_1(\alpha), P_2(\alpha)$ are equal, we get

$$(a-b)\alpha^2 = (b-c)\alpha + (c-a),$$

which gives

$$(a-b)(\alpha^2+1) = (b-c)(\alpha-1).$$

Similarly, using $P_2(\alpha) = P_3(\alpha)$, we obtain

$$(b-c)(\alpha^2+1) = (c-a)(\alpha-1).$$

If $\alpha = 1$, then it follows that $a = b = c$. Henceforth, let us assume that $\alpha \neq 1$. Then the above yields

$$(a-b)(c-a) = (b-c)^2.$$

Using a similar argument as above, it follows that

$$(b-c)(a-b) = (c-a)^2, (c-a)(b-c) = (a-b)^2.$$

Adding these equations, we obtain

$$a^2 + b^2 + c^2 = ab + bc + ca.$$

Since a, b, c are real, it follows that $a = b = c$. This completes the proof. ■

Example 1.7 (India RMO 2012f P1)**.** Find nonzero real numbers a, b such that $x^2 + ax + b, x^2 + x + ab, ax^2 + x + b$ are three distinct polynomials with a common root.

Solution 7. Let a, b be real numbers such that $x^2+ax+b, x^2+x+ab, ax^2+x+b$ are three distinct polynomials with a common root $\alpha \in \mathbb{C}$. We obtain

$$\alpha^2 + a\alpha + b = \alpha^2 + \alpha + ab = a\alpha^2 + \alpha + b = 0,$$

which gives $a\alpha + b = \alpha + ab$, that is, $(a - 1)(\alpha - b) = 0$. Since $x^2 + ax + b, x^2 + x + ab, ax^2 + x + b$ are distinct, it follows that $a \neq 1$. This shows that $\alpha = b$. Since the polynomials $x^2 + x + ab, ax^2 + x + b$ vanish at $x = \alpha = b$, we obtain

$$b(a + b + 1) = b(ab + 2) = 0.$$

Using b is nonzero, we get $a + b + 1 = ab + 2 = 0$. Note that $1 + ab - a - b = 0$. Since $a \neq 1$, we obtain $b = 1$, which combined with $ab + 2 = 0$ implies that $a = -2$.

Also note that for $a = -2, b = 1$, the given polynomials are equal to

$$x^2 - 2x + 1, x^2 + x - 2, -2x^2 + x + 1,$$

which are all distinct and they vanish at $x = 1$.

We conclude that precisely for $(a, b) = (-2, 1)$, the given polynomials are all distinct and have a common root. ■

Example 1.8 (India RMO 2015d P2)**.** Let $P(x) = x^2 + ax + b$ be a quadratic polynomial with real coefficients. Suppose there are real numbers s is not equal to t such that $P(s) = t$ and $P(t) = s$. Prove that $b - st$ is a root of the equation $x^2 + ax + b - st = 0$.

Solution 8. We have

$$s^2 + as + b = t, t^2 + at + b = s.$$

Taking their difference, we obtain $(s - t)(s + t + a + 1) = 0$, which gives $s + t + a + 1 = 0$ since $s \neq t$. Using the above, we obtain

$$s(s^2 + as + b) - t(t^2 + at + b) = 0,$$

or equivalently,

$$(s-t)(b+a(s+t)+s^2+st+t^2)=0.$$

Combining the above with $s+t+a+1=0$ and $s\neq t$, we obtain

$$b-(s+t)-st=0.$$

Note that

$$\begin{aligned}(b-st)^2+a(b-st)+b-st &= (b-st)(b-st+a+1)\\ &= (b-st)(s+t+a+1)\\ &= 0.\end{aligned}$$

This completes the proof. ■

Example 1.9 (India RMO 2015a P2)**.** Let $P(x)=x^2+ax+b$ be a quadratic polynomial where a, b are real numbers. Suppose $\langle P(-1)^2, P(0)^2, P(1)^2\rangle$ is an AP of positive integers. Prove that a, b are integers.

Solution 9. Note that $P(-1)^2, P(0)^2, P(1)^2$ are equal to

$$1+a^2+b^2-2a+2b-2ab, b^2, 1+a^2+b^2+2a+2b+2ab$$

respectively. Since they form an arithmetic progression, we obtain

$$1+a^2+b^2-2a+2b-2ab+1+a^2+b^2+2a+2b+2ab=2b^2,$$

or equivalently, $a^2+2b+1=0$. It follows that

$$b^2-2a-2ab, b^2, b^2+2a+2ab$$

form an arithmetic progression of positive integers. Note that

$$\begin{aligned}(2a+2ab)^2 &= 4a^2(b+1)^2\\ &= -4(2b+1)(b+1)^2\\ &= -4(2b^3+5b^2+4b+1)\\ &= -4((2b^2+4)b+5b^2+1).\end{aligned}$$

Since b^2 is an integer, it follows that b is rational number. Since b is rational and b^2 is an integer, it follows that b is an integer. Using $a^2+2b+1=0$, it follows that a^2 is an integer. Moreover, if $b=-1$, then a is an integer. If $b\neq -1$, then using that $2a+2ab$ is an integer, we obtain a is rational. Since a^2 is an integer and a is rational, it follows that a is an integer. This completes the proof. ■

Example 1.10 (India RMO 2015b P2)**.** Let $P(x) = x^2 + ax + b$ be a quadratic polynomial where a is real and $b \neq 2$, is rational. Suppose $P(0)^2, P(1)^2, P(2)^2$ are integers, prove that a and b are integers.

Solution 10. Since b is rational and $P(0)^2 = b^2$ is an integer, it follows that b is an integer. Note that

$$\begin{aligned} P(1)^2 &= (1 + a + b)^2 \\ &= 1 + a^2 + b^2 + 2a + 2b + 2ab, \\ P(2)^2 &= (4 + 2a + b)^2 \\ &= 16 + 4a^2 + b^2 + 16a + 8b + 4ab. \end{aligned}$$

Since b is an integer, the given conditions imply that $a^2+2a+2ab, 4a^2+16a+4ab$ are integers. This shows that

$$\begin{aligned} 4a^2 + 16a + 4ab - 2(a^2 + 2a + 2ab) &= 2a^2 + 12a, \\ 4a^2 + 16a + 4ab - 4(a^2 + 2a + 2ab) &= 8a - 4ab \end{aligned}$$

are integers. Since $b \neq 2$ and b is an integer, it follows that a is a rational number. Combining this with the fact that $2a^2 + 12a$ is rational, it follows that a is equal to $\frac{n}{2}$ for some integer n. Indeed, write $a = \frac{x}{y}$ where x, y are integers with $y \geq 1$ and $\gcd(x, y) = 1$. Note that $2\frac{x^2}{y} + 12x$ is an integer. Since x and y are relatively prime, this implies that y divides 2. Consequently, a is equal to $\frac{n}{2}$ for some integer n. Using that $2a^2 + 12a$ is an integer, we get that $\frac{n^2}{2}$ is also an integer. This shows that n is even, and hence a is an integer. This completes the proof. ∎

Example 1.11 (India RMO 2015e P2)**.** Let $P_1(x) = x^2 + a_1x + b_1$ and $P_2(x) = x^2 + a_2x + b_2$ be two quadratic polynomials with integer coefficients. Suppose $a_1 \neq a_2$ and there exist integers $m \neq n$ such that $P_1(m) = P_2(n), P_2(m) = P_1(n)$. Prove that $a_1 - a_2$ is even.

Solution 11. Using $P_1(m) = P_2(n)$, we get

$$m^2 + a_1m + b_1 = n^2 + a_2n + b_2,$$

that is,

$$(m^2 - n^2) + (a_1m - a_2n) + b_1 - b_2 = 0.$$

Similarly, using $P_1(n) = P_2(m)$, we get

$$(n^2 - m^2) + (a_1n - a_2m) + b_1 - b_2 = 0.$$

This yields

$$2(m^2 - n^2) + (a_1 + a_2)(m - n) = 0.$$

Since $m \neq n$, we get $2(m + n) + a_1 + a_2 = 0$. It follows that $a_1 + a_2$ is even, and hence, so is $a_1 + a_2 - 2a_2 = a_1 - a_2$. ∎

Example 1.12 (India RMO 2013c P4)**.** A polynomial is called a *Fermat polynomial* if it can be written as the sum of squares of two polynomials with integer coefficients. Suppose that $f(x)$ is a Fermat polynomial such that $f(0) = 1000$. Prove that $f(x) + 2x$ is not a Fermat polynomial.

> **Summary** — Assume that $f(x) + 2x$ is a Fermat polynomial. Write each of the polynomials $f(x), f(x) + 2x$ as a sum of the squares of two polynomials with integer coefficients. Reducing modulo 4, conclude that the constant terms of these four polynomials are even. Next, reduce modulo x^2, and compare the coefficients of x to obtain a contradiction.

Solution 12. On the contrary, let us assume that $f(x) + 2x$ is a Fermat polynomial.

Since $f(x)$ is a Fermat polynomial, it follows that there exist polynomials $P(x)$ and $Q(x)$ with integer coefficients such that $f(x) = P(x)^2 + Q(x)^2$. This gives $P(0)^2 + Q(0)^2 = f(0) = 1000$. Since a perfect square leaves one of $0, 1$ as a remainder upon division by 4, it follows that $P(0), Q(0)$ are even. Consequently, the coefficient of x in the polynomial $f(x) = P(x)^2 + Q(x)^2$ is a multiple of 4.

Since $f(x) + 2x$ is a Fermat polynomial, there exist polynomialss $R(x)$ and $S(x)$ with integer coefficients such that $f(x) + 2x = R(x)^2 + S(x)^2$. Using a similar argugemt as above, it follows that the coefficient of x in the polynomial $f(x) + 2x = R(x)^2 + S(x)^2$ is a multiple of 4.

Note that

$$2x = P(x)^2 + Q(x)^2 - R(x)^2 - S(x)^2,$$

and the coefficient of x in $P(x)^2 + Q(x)^2 - R(x)^2 - S(x)^2$ is a multiple of 4. This contradicts the assumption that $f(x) + 2x$ is a Fermat polynomial.

This proves that $f(x) + 2x$ is not a Fermat polynomial. ∎

Example 1.13 (India RMO 2023a P3)**.** Let $f(x)$ be a polynomial with real coefficients of degree 2. Suppose that for some pairwise distinct **nonzero** real numbers , a, b, c we have:

$$f(a) = bc, f(b) = ac, f(c) = ab$$

Dertermine $f(a + b + c)$ in terms of a, b, c.

Solution 13. Note that cubic polynomial $xf(x) - abc$ has three roots a, b, c, and hence

$$xf(x) - abc = \lambda(x - a)(x - b)(x - c)$$

holds for some real number λ. Substituting $x = 0$ in the above, and using that a, b, c are nonzero, it follows that $\lambda = 1$. This gives $f(x) = x^2 - (a + b + c)x + ab + bc + ca$. We obtain that

$$f(a + b + c) = ab + bc + ca.$$

∎

Example 1.14 (India RMO 2024a P2)**.** Show that there do not exist non-zero real numbers a, b, c such that the following statements hold simultaneously:

1. the equation $ax^2 + bx + c = 0$ has two distinct roots x_1, x_2;
2. the equation $bx^2 + cx + a = 0$ has two distinct roots x_2, x_3;
3. the equation $cx^2 + ax + b = 0$ has two distinct roots x_3, x_1.

(Note that x_1, x_2, x_3 may be real or complex numbers.)

Solution 14. On the contrary, let us assume that there exist nonzero real numbers a, b, c such that the given condition holds. Note that

$$ax_2^2 + bx_2 + c = 0, bx_2^2 + cx_2 + a = 0$$

hold. Eliminating x_2^2, we obtain

$$(b^2 - ca)x_2 = a^2 - bc.$$

Also note that

$$(a^2 - bc)x_2^2 = (c^2 - ab)x_2$$

holds. Since $c \neq 0$, it follows that $x_2 \neq 0$, and hence

$$(a^2 - bc)x_2 = c^2 - ab.$$

Combining the above, we obtain

$$(a^2 - bc)^2 = (b^2 - ca)(c^2 - ab),$$

which can be simplified to

$$a(a^3 + b^3 + c^3 - 3abc) = 0.$$

Since a is nonzero, it implies that $a^3 + b^3 + c^3 = 3abc$, and hence $a + b + c = 0$ or $a = b = c$.

Note that if a, b, c satisfy $a = b = c$, then each of x_1, x_2, x_3 is equal to one of ω, ω^2, and hence, two of x_1, x_2, x_3 are equal, which is impossible by our assumption.

Suppose a, b, c satisfy $a + b + c = 0$. Note that if some two of a, b, c are equal, then one of the polynomials $ax^2 + bx + c, bx^2 + cx + a, cx^2 + ax + b$ does not have distinct roots. Note that if the roots of $ax^2 + bx + c$ are equal, then $1 = -b/a - 1$ holds, which implies that $b = -2a$, and hence $a = c$, which is impossible. Hence, the roots of $ax^2 + bx + c$ are distinct. By similar arguments, it follows that each of $bx^2 + cx + a, cx^2 + ax + b$ possesses distinct roots. If the common root of $ax^2 + bx + c, bx^2 + cx + a$ is equal to 1, then considering the common root of $bx^2 + cx + a, cx^2 + ax + b$, we obtain $c/b = a/c$, which gives $c^2 = ab$, and this yields $(a + b)^2 = ab$, which is impossible since a, b are nonzero real numbers. If the common root of $ax^2 + bx + c, bx^2 + cx + a$ is not equal to 1, then we obtain $b/a = c/b$, which implies that $(c + a)^2 = b^2 = ac$, which is impossible since a, c are nonzero real numbers.

The completes the proof. ∎

References

[**Che25**] Evan Chen. *The OTIS Excerpts*. Available at https://web.evanchen.cc/excerpts.html. 2025, pp. vi+289