Polynomials

MOPSS

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Mathematics Olympiad

Problem Solving Sessions

MOPSS

Department of Mathematics
IISER Bhopal

https://jpsaha.github.io/MOTP/MOPSS/

Suggested readings

- Evan Chen's advice On reading solutions, available at https://blog.evanchen.cc/2017/03/06/on-reading-solutions/.
- Evan Chen's Advice for writing proofs/Remarks on English, available at https://web.evanchen.cc/handouts/english/english.pdf.
- Notes on proofs by Evan Chen from OTIS Excerpts [**Che25**, Chapter 1].
- Tips for writing up solutions by Edward Barbeau, available at https://www.math.utoronto.ca/barbeau/writingup.pdf.
- Evan Chen discusses why math olympiads are a valuable experience for high schoolers in the post on Lessons from math olympiads, available at https://blog.evanchen.cc/2018/01/05/lessons-from-math-olympiads/.

List of problems and examples

§1 Polynomials

For further problems, we refer to [**Goy21**].

§1.1 Warm up

Example 1.1. Factorize the polynomial $x^8 + x^4 + 1$ into factors of at most the second degree.

> **Summary** — Expressing an expression as a difference of two squares yields a factorization.

Solution 1. Note that

$$x^8 + x^4 + 1$$

$$
\begin{aligned}
&= x^8 + 2x^4 + 1 - x^4 \\
&= (x^4+1)^4 - (x^2)^2 \\
&= (x^4 - x^2 + 1)(x^4 + 1 + x^2) \\
&= (x^4 + 2x^2 + 1 - 3x^2)(x^4 + 2x^2 + 1 - x^2) \\
&= ((x^2+1)^2 - (\sqrt{3}x)^2)((x^2+1)^2 - x^2) \\
&= (x^2 - \sqrt{3}x + 1)(x^2 + \sqrt{3}x + 1)(x^2 - x + 1)(x^2 + x + 1).
\end{aligned}
$$

■

Example 1.2. Show that

$$2a^2b^2 + 2b^2c^2 + 2c^2a^2 - a^4 - b^4 - c^4 = (a+b+c)(a+b-c)(b+c-a)(c+a-b).$$

Summary — Complete the squares.

Walkthrough —

(a) Try to see what would happen if we were allowed to change the signs!

(b) Change the signs and consider

$$a^4 + b^4 + c^4 - 2a^2b^2 - 2b^2c^2 - 2c^2a^2.$$

This is “almost” $(a^2 - b^2 - c^2)^2$!! To be precise

$$(a^2 - b^2 - c^2)^2 = a^4 + b^4 + c^4 - 2a^2b^2 + 2b^2c^2 - 2c^2a^2.$$

(c) Let us continue with the above, and write $2a^2b^2+2b^2c^2+2c^2a^2-a^4-b^4-c^4$ in terms $(a^2 - b^2 - c^2)^2$ as follows.

$$
\begin{aligned}
&2a^2b^2 + 2b^2c^2 + 2c^2a^2 - a^4 - b^4 - c^4 \\
&= 4b^2c^2 - (a^4 + b^4 + c^4 - 2a^2b^2 + 2b^2c^2 - 2c^2a^2).
\end{aligned}
$$

(d) Does the above help?

Solution 2. Note that

$$
\begin{aligned}
&2a^2b^2 + 2b^2c^2 + 2c^2a^2 - a^4 - b^4 - c^4 \\
&= 4b^2c^2 - (a^4 + b^4 + c^4 - 2a^2b^2 + 2b^2c^2 - 2c^2a^2) \\
&= (2bc)^2 - (a^2 - b^2 - c^2)^2 \\
&= (2bc - (a^2 - b^2 - c^2))(2bc + a^2 - b^2 - c^2) \\
&= (2bc + b^2 + c^2 - a^2)(a^2 - (b^2 + c^2 - 2bc)) \\
&= ((b+c)^2 - a^2)(a^2 - (b-c)^2)
\end{aligned}
$$

$$
\begin{aligned}
&= (a+b+c)(b+c-a)(a+b-c)(a-b+c)\\
&= (a+b+c)(a+b-c)(b+c-a)(c+a-b).
\end{aligned}
$$

■

Solution 3. Write[1]

$$
\begin{aligned}
a &= y+z,\\
b &= z+x,\\
c &= x+y.
\end{aligned}
$$

Note that

$$
\begin{aligned}
&2a^2b^2 + 2b^2c^2 + 2c^2a^2 - a^4 - b^4 - c^4\\
&= 2(y+z)^2(z+x)^2 + 2(z+x)^2(x+y)^2 + 2(x+y)^2(y+z)^2\\
&- (y+z)^4 - (z+x)^4 - (x+y)^4\\
&= 2(z^2+2yz+y^2)(z^2+2zx+x^2)\\
&+ 2(x^2+2zx+z^2)(x^2+2xy+y^2)\\
&+ 2(y^2+2xy+x^2)(y^2+2yz+z^2)\\
&- (y+z)^4 - (z+x)^4 - (x+y)^4\\
&= 2\sum_{\text{cyc}} \left(x^4 + x^2(y^2+z^2+2x(y+z)) + yz(2x+y)(2x+z)\right) - \sum_{\text{cyc}}(x+y)^4\\
&= 2\sum_{\text{cyc}} \left(x^4 + x^2y^2 + z^2x^2 + 2x^3(y+z) + 4x^2yz + 2xyz(y+z) + y^2z^2\right)\\
&- \sum_{\text{cyc}}(x+y)^4\\
&= 2(x^4+y^4+z^4) + 6(x^2y^2+y^2z^2+z^2x^2) + 16xyz(x+y+z) + 4\sum_{\text{cyc}} x^3(y+z)\\
&- \sum_{\text{cyc}}(x+y)^4\\
&= 2(x^4+y^4+z^4) + 6(x^2y^2+y^2z^2+z^2x^2) + 16xyz(x+y+z) + 4\sum_{\text{cyc}} x^3(y+z)
\end{aligned}
$$

[1] It is good ask the following simple and innocent question: how can one **write** a, b, c as stated above? Does it mean that given any three real numbers a, b, c, one can find real numbers x, y, z such that $a = y+z, b = z+x, c = x+y$? A crucial point to note is that one very often deals with indeterminates (aka variables) instead of real numbers. In the above, a, b, c could be indeterminates instead of being real numbers! What do we do in that case? Is it so that there are **indeterminates** x, y, z such that the six indeterminates a, b, c, x, y, z satisfy $a = y+z, b = z+x, c = x+y$? As of now, **let's not worry about any of these!**. Just keep in mind the message that at times, **we need to be quite careful about what we do!**

$$- \sum_{\text{cyc}} (x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4)$$
$$= 16xyz(x + y + z)$$
$$= (a + b + c)(a + b - c)(b + c - a)(c + a - b).$$

■

Remark. Please go through the footnote next to the word **Write** from the above solution. This footnote would explain that just writing **Write** $a = y + z, b = z + x, c = x + y$ requires more care! To address this issue, replace **Write** $a = y + z, b = z + x, c = x + y$ in the above solution by the following.

Consider the real numbers x, y, z defined by

$$x = \frac{1}{2}(b + c - a),$$
$$y = \frac{1}{2}(c + a - b),$$
$$z = \frac{1}{2}(a + b - c).$$

Note that

$$a = y + z, b = z + x, c = x + y$$

holds.

Example 1.3. Find numbers a, b, c, d for which the equation

$$\frac{2x - 7}{4x^2 + 16x + 15} = \frac{a}{x + c} + \frac{b}{x + d}$$

would be an identity.

Walkthrough — Factorize the denominator into linear factors. Then expressing the numerator as a linear combination of those factors would provide such an identity.

Solution 4. Note that

$$\frac{2x - 7}{4x^2 + 16x + 15} = \frac{2x - 7}{(2x + 3)(2x + 5)}.$$

Hence, if $2x - 7$ can be expressed as

$$p(2x + 3) + q(2x + 5),$$

then $\frac{2x-7}{4x^2+16x+15}$ can be expressed as a sum of two fractions, each having a constant in the numerator and a linear polynomial in the denominator.

One way to find if there are any such p, q, is to assume first that there are such real numbers p and q such that

$$2x - 7 = p(2x + 3) + q(2x + 5)$$

holds[2]. Substituting $x = -\frac{5}{2}$, we obtain $-2p = -12$, which gives $p = 6$. Next, substituting $x = -\frac{3}{2}$, we obtain $2q = -10$, which implies $q = -5$.

Note that

$$6(2x + 3) + (-5)(2x + 5) = 12x + 18 - 10x - 25 = 2x - 7$$

holds[3]. Using it, we obtain

$$\begin{aligned} \frac{2x-7}{4x^2+16x+15} &= \frac{2x-7}{(2x+3)(2x+5)} \\ &= \frac{6(2x+3)+(-5)(2x+5)}{(2x+3)(2x+5)} \\ &= \frac{6}{2x+5} - \frac{5}{2x+3} \\ &= \frac{3}{x+\frac{5}{2}} - \frac{\frac{5}{2}}{x+\frac{3}{2}}. \end{aligned}$$

Hence, we may take

$$a = 3, b = -\frac{5}{2}, c = \frac{5}{2}, d = \frac{3}{2}.$$

■

Exercise 1.4. Are there other choices for a, b, c, d for which the identify would hold?

Example 1.5. Let n be a positive integer. Show that

$$x^n - 1 = (x - 1)(x^{n-1} + x^{n-2} + \cdots + x + 1).$$

Solution 5. Note that

$$(x - 1)(x^{n-1} + x^{n-2} + \cdots + x + 1)$$

[2] and try to see what conditions get imposed on p, q. It may happen that the conditions that get imposed, may suggest that there are no such p, q. However, it may also happen that we would be able to find out which p, q would work!

[3] It should be noted that p, q were assumed to exist such that $p(2x+3)+q(2x+5) = 2x-7$ holds. Under this hypothesis, we obtained $p = 6, q = -5$. At this point, we cannot immediately conclude that $6(2x + 3) + (-5)(2x + 5) = 2x - 7$ holds (unless we verify it), because if we do so, then we would do it under the same hypothesis.

- Even then, what would go wrong with that?
- Can a hypothesis (possibly combined with some of its consequences) be a justification for itself to hold? Think about this point.

$$
\begin{aligned}
&= x(x^{n-1} + x^{n-2} + \cdots + x + 1) - (x^{n-1} + x^{n-2} + \cdots + x + 1) \\
&= x^n + x^{n-1} + \cdots + x^2 + x - (x^{n-1} + x^{n-2} + \cdots + x + 1) \\
&= x^n - 1.
\end{aligned}
$$

■

The exercise below relies on Example 1.5.

Example 1.6 (Moscow MO 2015 Grade 9 P6)**.** Do there exist two polynomials with integer coefficients such that each of them has a coefficient with absolute value exceeding 2015, but no coefficient of their product has absolute value exceeding 1?

Summary — Try to come up with enough polynomials $g_1(x), g_2(x), g_3(x), \ldots$ and $h_1(x), h_2(x), h_3(x), \ldots$ such that each of the products $g_1 g_2 g_3 \ldots$ and $h_1 h_2 h_3 \ldots$ have at least one coefficient which is **large in absolute value**, and all the coefficients of the product $(g_1 g_2 g_3 \ldots)(h_1 h_2 h_3 \ldots)$ are at most 1 in absolute value.

Walkthrough —

(a) Try to come up with a polynomial $P(x)$ whose coefficients are at most 1 in absolute value, and it can be written as a product of enough factors (say $f_1(x), f_2(x), \ldots$) such that each of such factor $f_i(x)$ admits a decomposition into the product of two polynomials $g_i(x)$ and $h_i(x)$.

(b) Can you make sure that the product of the g_i's, and the product of the h_i's have to have at least one large coefficient?

(c) For instance, would taking $g_1(x) = g_2(x) = g_3(x) = \cdots = 1 - x$ work for some suitable choice of $h_1(x), h_2(x), \ldots$?

(d) Does taking

$$
\begin{aligned}
h_1(x) &= 1 + x, \\
h_2(x) &= 1 + x + x^2, \\
h_3(x) &= 1 + x + x^2 + x^3,
\end{aligned}
$$

etc. work?

(e) Note that the product of enough g_i's would have a large coefficient (namely, the coefficient of the second largest power of x). On the other hand, the product of enough h_i's would have a large coefficient (namely, the coefficient of the power of x).

(f) What can be said about the absolute value of the coefficients of the product of these two products?

The above seems to work except that having a control on the coefficients of the product $(g_1 g_2 g_3 \ldots)(h_1 h_2 h_3 \ldots)$ seems hard[4].

[4] Is it because it fails?

Solution 6. Consider the polynomial

$$P(x) = (1-x)(1-x^2)(1-x^4)(1-x^8)\cdots(1-x^{2^{2016}}).$$

Since

$$1 + 2 + 2^2 + 2^3 + \cdots + 2^{n-1} < 2^n,$$

it follows that the coefficients of $P(x)$ are at most 1 in absolute value. Note that

$$P(x) = Q(x)R(x)$$

holds where

$$Q(x) = (1-x)^{2017},$$
$$R(x) = (1+x)(1+x+x^2+x^3)\cdots(1+x+x^2+\cdots+x^{2^{2016}-1}).$$

The coefficient of x^{2016} in $Q(x)$ is equal to 2017, and the coefficient of x in $R(x)$ is equal to 2016. This completes the proof. ■

§1.2 Division algorithm

Example 1.7. Prove that the polynomial $x^{44} + x^{33} + x^{22} + x^{11} + 1$ is divisible by the polynomial $x^4 + x^3 + x^2 + x + 1$.

Solution 7. Note that

$$\begin{aligned}
&x^{44} + x^{33} + x^{22} + x^{11} + 1 \\
&= x^{40} \cdot x^4 + x^{30} \cdot x^3 + x^{20} \cdot x^2 + x^{10} \cdot x + 1 \\
&= (x^{40} - 1)x^4 + (x^{30} - 1)x^3 + (x^{20} - 1)x^2 + (x^{10} - 1)x \\
&\quad + x^4 + x^3 + x^2 + x + 1.
\end{aligned}$$

Hence, to prove that the polynomial $x^{44} + x^{33} + x^{22} + x^{11} + 1$ is divisible by $x^4 + x^3 + x^2 + x + 1$, it suffices to show that $x^4 + x^3 + x^2 + x + 1$ divides the polynomials

$$x^{40} - 1, x^{30} - 1, x^{20} - 1, x^{10} - 1.$$

Since

$$\begin{aligned}
x^5 - 1 &= (x-1)(x^4 + x^3 + x^2 + x + 1), \\
x^{10} - 1 &= (x^5 - 1)(x^5 + 1),
\end{aligned}$$

it follows that $x^4 + x^3 + x^2 + x + 1$ divides $x^{10} - 1$. Moreover, the polynomial x^{10-1} divides all of

$$x^{40} - 1, x^{30} - 1, x^{20} - 1, x^{10} - 1.$$

Hence, $x^4 + x^3 + x^2 + x + 1$ divides the polynomials

$$x^{40} - 1, x^{30} - 1, x^{20} - 1, x^{10} - 1.$$

■

Exercise 1.8. Show that the polynomial $x^{580}+x^{390}+x^{326}+x^{262}+x^{198}+x^{134}+1$ is divisible by $x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$.

Example 1.9. Determine the remainder obtained upon dividing x^{100} by $x^2 - 3x + 2$.

Solution 8. Let $q(x)$ (resp. $r(x)$) denote the quotient (resp. the remainder) obtained upon dividing x^{100} by x^2-3x+2. Note that $r(x)$ is a linear polynomial, i.e. $r(x) = ax + b$ for some real numbers a, b. Then we have

$$x^{100} = q(x)(x^2 - 3x + 2) + r(x).$$

Substituting $x = 1$, it yields

$$1 = r(1) = a + b.$$

Similarly, substituting $x = 2$, it gives

$$2^{100} = r(2) = 2a + b.$$

This shows that

$$a = 2^{100} - 1, \quad b = 1 - a = 2 - 2^{100}.$$

Hence, the remainder obtained upon dividing x^{100} by $x^2 - 3x + 2$ is equal to

$$(2^{100} - 1)x + 2 - 2^{100}.$$

■

Example 1.10. Factorize the polynomials

$$x(y - z)^3 + y(z - x)^3 + z(x - y)^3, \quad xy(x^2 - y^2) + yz(y^2 - z^2) + zx(z^2 - x^2)$$

into products of linear polynomials.

§1.3 Even and odd polynomials

Example 1.11 (Moscow MO 1946 Grades 7–8 P5)**.** Prove that after completing the multiplication and collecting the terms

$$(1 - x + x^2 - x^3 + \cdots - x^{99} + x^{100})(1 + x + x^2 + \cdots + x^{99} + x^{100})$$

has no monomials of odd degree.

Summary — What happens if x is replaced by $-x$?

Solution 9. Let $P(x)$ denote the polynomial

$$(1 - x + x^2 - x^3 + \cdots - x^{99} + x^{100})(1 + x + x^2 + \cdots + x^{99} + x^{100}).$$

Note that $P(x) = P(-x)$. By the Claim below, it follows that $P(x)$ has no monomials of odd degree.

Claim — Let $Q(x)$ be a polynomial satisfying $Q(x) = Q(-x)$. Then $Q(x)$ has no monomials of odd degree.

Proof of the Claim. Note that

$$Q(x) = \frac{Q(x) + Q(-x)}{2} + \frac{Q(x) - Q(-x)}{2}$$

holds. Using $Q(x) = Q(-x)$, it follows that $Q(x) = \frac{Q(x)+Q(-x)}{2}$. Consequently, $Q(x)$ has no monomials of odd degree. □

■

Remark. The above decomposition of $Q(x)$ is a special case of general phenomena[a].

[a] Can you think of a few? Which **general phenomena** is referred to?!

Remark. The above solution is more elegant, and less cumbersome. Moreover, it also highlights the underlying reason, whereas the solution below obscures the conceptual viewpoint.

Solution 10. One can multiply the polynomials to note that

$$\begin{aligned}
&1 - x + x^2 - x^3 + \cdots - x^{99} + x^{100} \\
&= 1 - x + x^2(1 - x) + x^4(1 - x) + x^6(1 - x) + \cdots + x^{98}(1 - x) + x^{100} \\
&= (1 - x)(1 + x^2 + x^4 + x^6 + \cdots + x^{98}) + x^{100}.
\end{aligned}$$

Using this, we obtain

$$\begin{aligned}
&(1 - x + x^2 - x^3 + \cdots - x^{99} + x^{100})(1 + x + x^2 + \cdots + x^{99} + x^{100}) \\
&= \big((1 - x)(1 + x^2 + x^4 + x^6 + \cdots + x^{98}) + x^{100}\big)(1 + x + x^2 + \cdots + x^{99} + x^{100}) \\
&= (1 - x)(1 + x^2 + x^4 + x^6 + \cdots + x^{98})(1 + x + x^2 + \cdots + x^{99} + x^{100}) \\
&\quad + x^{100}(1 + x + x^2 + \cdots + x^{99} + x^{100}) \\
&= (1 + x^2 + x^4 + x^6 + \cdots + x^{98})(1 - x^{101}) \\
&\quad + x^{100}(1 + x + x^2 + \cdots + x^{99} + x^{100})
\end{aligned}$$

$$
\begin{aligned}
&= (1 + x^2 + x^4 + x^6 + \cdots + x^{98})(1 - x^{101}) \\
&\quad + x^{100}(x + x^3 + x^5 + \cdots + x^{99}) \\
&\quad + x^{100}(1 + x^2 + x^4 + \cdots + x^{98} + x^{100}) \\
&= (1 + x^2 + x^4 + x^6 + \cdots + x^{98})(1 - x^{101}) \\
&\quad + x^{101}(1 + x^2 + x^4 + x^6 + \cdots + x^{98}) \\
&\quad + x^{100}(1 + x^2 + x^4 + \cdots + x^{98} + x^{100}) \\
&= 1 + x^2 + x^4 + x^6 + \cdots + x^{98} + x^{100}(1 + x^2 + x^4 + \cdots + x^{98} + x^{100}),
\end{aligned}
$$

which has no monomial of odd degree. ■

The following exercise is quite similar to the Claim proved in the solution to Example 1.11.

Exercise 1.12. Let $Q(x)$ be a polynomial satisfying $Q(x) = -Q(-x)$. Then $Q(x)$ has no monomials of even degree.

Example 1.13. Let n be an even positive integer, and let $p(x)$ be a polynomial of degree n such that $p(k) = p(-k)$ for $k = 1, 2, \ldots, n$. Prove that there is a polynomial $q(x)$ such that $p(x) = q(x^2)$.

Walkthrough — Note that the polynomial $p(x) - p(-x)$ has degree $< n$ because n is even. Observe that it has at least n roots.

Remark. What would happen if n is not assumed to be even?

Example 1.14 (Tournament of Towns, Spring 2014, Senior, A Level, P7 by D. A. Zvonkin)**.** Consider a polynomial $P(x)$ such that

$$P(0) = 1, \quad (P(x))^2 = 1 + x + x^{100}Q(x),$$

where $Q(x)$ is also a polynomial. Prove that in the polynomial $(P(x) + 1)^{100}$, the coefficient of x^{99} is zero.

Solution 11. Note that

$$(P(x) + 1)^{100} + (1 - P(x))^{100}$$

is a polynomial in $P(x)^2$ of degree 50. Given three polynomials $f(x), g(x), h(x)$ having complex coefficients, with $h(x) \neq 0$, we say that $f(x)$ *is congruent to* $g(x)$ *modulo* $h(x)$ if $h(x)$ divides $f(x) - g(x)$, that is, $f(x) - g(x)$ is the product of $h(x)$ and a polynomial in x with complex coefficients. Since $P(x)^2$ is congruent to $1 + x$ modulo x^{100}, it follows that $(P(x) + 1)^{100} + (1 - P(x))^{100}$ is congruent to a polynomial of degree 50 in $1 + x$ modulo x^{100}. Using that $P(x) \equiv 1 \bmod x$, we obtain that $(P(x) + 1)^{100}$ is congruent to a polynomial of degree 50 in $1 + x$ modulo x^{100}. This shows that the coefficient of x^{99} in $(P(x) + 1)^{100}$ is zero. ■

§1.4 Factorization and roots

Example 1.15. Let a, b, c be three distinct real numbers. Show that

$$\frac{(a-x)(b-x)}{(a-c)(b-c)} + \frac{(b-x)(c-x)}{(b-a)(c-a)} + \frac{(c-x)(a-x)}{(c-b)(a-b)} = 1.$$

> **Walkthrough** — Can a polynomial having degree at most two admit more than two distinct roots?

Example 1.16 (USAMO 1975 P3). [**GA17**, Problem 151] A polynomial $P(x)$ of degree n satisfies

$$P(k) = \frac{k}{k+1} \qquad \text{for } k = 0, 1, 2, \dots, n.$$

Find $P(n+1)$.

Solution 12. Note that $xP(x+1) - x$ is a polynomial of degree $n+1$, and it vanishes at the $n+1$ integers $0, 1, 2, \dots, n$. It follows that

$$(x+1)P(x) - x = cx(x-1)(x-2)\dots(x-n)$$

for some nonzero real number c. Substituting $x = -1$ yields

$$1 = (-1)^{n+1}c(n+1)!,$$

which gives $c = \frac{(-1)^{n+1}}{(n+1)!}$. This implies that

$$(n+2)P(n+1) = n+1+(-1)^{n+1},$$

and consequently,

$$P(n+1) = \frac{n+1+(-1)^{n+1}}{n+2}.$$

■

Example 1.17. Let $P(x)$ be a polynomial of degree $\leq n$ having rational coefficients. Suppose $P(k) = \frac{1}{k}$ holds for $1 \leq k \leq n+1$. Determine $P(0)$.

Solution 13. Note that $xP(x) - 1$ is a polynomial of degree at most $n+1$, and it vanishes at $1, 2, \dots, n+1$. This implies that

$$xP(x) - 1 = c(x-1)(x-2)\dots(x-n-1)$$

holds for some rational number c. Substituting $x = 0$, we obtain

$$c = \frac{(-1)^n}{(n+1)!}.$$

Differentiating and dividing by $xP(x) - 1$, we obtain

$$\frac{xP'(x) + P(x)}{xP(x) - 1} = \frac{1}{x-1} + \frac{1}{x-2} + \cdots + \frac{1}{x-n-1},$$

which yields

$$P(0) = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}.$$

■

Example 1.18. Let $g(x)$ and $h(x)$ be polynomials with real coefficients such that

$$g(x)(x^2 - 3x + 2) = h(x)(x^2 + 3x + 2)$$

and $f(x) = g(x)h(x) + (x^4 - 5x^2 + 4)$. Prove that $f(x)$ has at least four real roots.

Solution 14. Note that $g(x)$ and $h(x)$ satisfy

$$g(x)(x-1)(x-2) = h(x)(x+1)(x+2),$$

which shows that

$$g(-1), g(-2), h(1), h(2)$$

are equal to 0. Also note that

$$\begin{aligned} x^4 - 5x^2 + 4 &= (x^2 - 1)(x^2 - 4) \\ &= (x-1)(x+1)(x+2)(x-2). \end{aligned}$$

Hence, the polynomials $g(x)h(x)$ and $x^4 - 5x^2 + 4$ vanish at $1, -1, 2, -2$. Consequently, f also vanishes at these four points. ■

Example 1.19. Let $P(x)$ be a polynomial with real coefficients such that $P(\sin\alpha) = P(\cos\alpha)$ for all $\alpha \in \mathbb{R}$. Show that $P(x) = Q(x^2 - x^4)$ for some polynomial $Q(x)$ with real coefficients.

Walkthrough —

(a) Show that $P(x) = P(-x)$ for any $-1 \le x \le 1$, and hence $P(x) = f(x^2)$.

(b) Deduce that $f(x) = f(1-x)$ for any $0 \le x \le 1$.

(c) Using induction or otherwise, prove that $f(x) = g(x - x^2)$ for some polynomial $g(x)$ with real coefficients.

Example 1.20. [WH96, Problem 27] Let $p_1, \ldots, p_n$ denote $n \ge 1$ distinct integers. Show that the polynomial

$$(x - p_1)^2(x - p_2)^2 \cdots (x - p_n)^2 + 1$$

cannot be expressed as the product of two non-constant polynomials with integral coefficients.

Solution 15. On the contrary, let us assume that the polynomial

$$P(x) := (x - p_1)^2(x - p_2)^2 \cdots (x - p_n)^2 + 1$$

can be expressed as the product of two non-constant polynomials $f(x), g(x)$ with integral coefficients.

Let us first establish the following Claims.

> **Claim** — Replacing f, g by $-f, -g$ respectively (if necessary), we may assume that f, g take positive values at all real arguments.

Proof of the Claim. Note that the polynomial $P(x)-1$ vanishes at $x = p_1, \dots, p_n$. Since the product of the leading coefficients of $f(x)$ and $g(x)$ is equal to the leading coefficient of $P(x)$, we may replace $f(x), g(x)$ by $-f(x), -g(x)$ respectively (if necessary) to assume that the leading coefficients of $f(x), g(x)$ are positive. Since $P = fg$ and P does not have a real root, it follows that the polynomials f, g do not have any real roots. At large enough real arguments, the polynomials f, g take positive values. Since f, g have no real roots, we conclude that they take positive values at all real arguments. □

> **Claim** — The polynomials f, g are of degree n. Moreover, these polynomials are equal.

Proof of the Claim. On the contrary, let us asssume that the degrees of f, g are not equal. Interchanging f, g if necessary, we assume that $\deg(f) < \deg(g)$. Since the sum of the degrees of f, g is equal to $2n$, it follows that $\deg(f) < n$.

For any $1 \leq i \leq n$, the integers $f(p_i), g(p_i)$ are equal to 1 or -1. Since f, g take positive values at all real arguments, we obtain $f(p_i) = 1$ for any $1 \leq i \leq n$. This shows that the polynomial $f - 1$ has at least n distinct roots. Using $\deg(f) < n$, we conclude that $f - 1$ is the zero polynomial, which is impossible since f is a non-constant polynomial. Therefore, the hyothesis that the degrees of f, g are not equal is not tenable. This completes the proof of the first part of the Claim.

Note that f, g are polynomials of degree n with equal leading coefficients. This shows that the polynomial $f(x) - g(x)$ has degree less than n and it vanishes at the n distinct points $p_1, \dots, p_n$. It follows that $f = g$.

□

Using the above Claim, note that

$$f(x)^2 - ((x - p_1)(x - p_2) \cdots (x - p_n))^2 = 1,$$

or equivalently,

$$\big(f(x) + (x - p_1)(x - p_2) \cdots (x - p_n)\big)\big(f(x) - (x - p_1)(x - p_2) \cdots (x - p_n)\big)$$

$$= 1,$$

which implies that the polynomials

$$f(x) + (x - p_1)(x - p_2) \cdots (x - p_n), f(x) - (x - p_1)(x - p_2) \cdots (x - p_n)$$

are constant polynomials, and both of them are equal. Consequently, the polynomial $(x-p_1)(x-p_2)\cdots(x-p_n)$ is the zero polynomial, which is impossible. This shows that the hypothesis that the given polynomial can be expressed as the product of two non-constant polynomials with integral coefficients is not tenable. This completes the proof. ■

Example 1.21 (cf. Moscow MO 1953 Grade 10, India BMath 2006 P3). [**AE11**, §1.7] Let n be a positive integer. Find the roots of the polynomial

$$P_n(X) = 1 + \frac{X}{1!} + \frac{X(X+1)}{2!} + \cdots + \frac{X(X+1)\cdots(X+n-1)}{n!}.$$

> **Walkthrough** — Note that $P_1(X) = X + 1$ has -1 as its root, $P_2(X) = \frac{1}{2}(X+1)(X+2)$ has $-1, -2$ as its roots. Check that
>
> $$P_3(X) = \frac{1}{3!}(X+1)(X+2)(X+3).$$
>
> What happens for general n?

Solution 16. We claim that the roots of $P_n(X)$ are $-1, -2, \ldots, -n$ for any integer $n \geq 1$. Note that the claim holds for $n = 1$. Suppose the claim holds for some integer n. Comparing leading coefficients, it follows that

$$P_n(X) = \frac{1}{n!}(X+1)(X+2)\cdots(X+n).$$

Observe that

$$\begin{aligned} P_{n+1}(X) &= P_n(X) + \frac{X(X+1)\cdots(X+n)}{(n+1)!} \\ &= \frac{1}{n!}(X+1)(X+2)\cdots(X+n) + \frac{X(X+1)\cdots(X+n)}{(n+1)!} \\ &= \frac{1}{(n+1)!}(X+1)(X+2)\cdots(X+n)(n+1+X). \end{aligned}$$

Hence the roots of P_{n+1} are $-1, -2, \ldots, -(n+1)$. The claim follows by induction. ■

Example 1.22. Show that any odd degree polynomial with real coefficients has at least one real root.

Example 1.23 (Putnam 1999 A2)**.** Show that for some fixed positive integer n, we can always express a polynomial with real coefficients which is nowhere negative as a sum of the squares of n polynomials.

Walkthrough —

(a) Show that the real roots of P have even multiplicity.

(b) Conclude that P can be expressed as a product of monic quadratic polynomials with real coefficients having nonreal roots, and even powers of linear polynomials with real coefficients.

(c) Show that a monic quadratic polynomial with real coefficients having nonreal roots is the sum of the squares of two polynomials with real coefficients.

Solution 17. Note that if P is a constant polynomial, then it is clear. Henceforth, let us assume that P is a nonconstant polynomial.

Claim — The polynomial P can be written as the product of polynomials, each of which can be expressed as the sum of the squares of two polynomials with real coefficients.

Proof of the Claim. Since P has real coefficients, it follows that if $\alpha \in \mathbb{C} \setminus \mathbb{R}$ is a root of P, then so is $\overline{\alpha}$. Thus, the nonreal complex roots of P form pairs of complex conjugates. Note that

$$(x-\alpha)(x-\overline{\alpha}) = (x - \operatorname{Re}(\alpha))^2 + \operatorname{Im}(\alpha)^2.$$

Decomposing P over the pairs of nonreal complex conjugate roots, and the real roots, it follows that P can be expressed as the product

$$cf(x) \prod_{a \in A} (x-a)^{m_a},$$

where c denote the leading coefficient of P, $f(x)$ denotes the product of (possibly no) quadratic polynomials of the form $(x-a)^2 + b^2$ with $a \in \mathbb{R}, b \in \mathbb{R} \setminus \{0\}$, and A denotes the set of real roots of P, and for an element $a \in A$, the multiplicity of a is denoted by m_a.

Evaluating P at a suitable real number (for instance, at $1 + \sum_{a \in A} a$ (resp. 1) if A is nonempty (resp. empty)), it follows that $c > 0$.

Let a be an element of A. Since A is finite, there exists a real number $\varepsilon > 0$ such that the interval $(a - \varepsilon, a + \varepsilon)$ contains no real roots of P other than a. If m_a were odd, then the sign of $P(x)$ would not remain constant as x ranges over in $(a-\varepsilon, a+\varepsilon) \setminus \{a\}$. Hence, it follows that m_a is even.

Since $c > 0$ amd m_a is even for any $a \in A$, the Claim follows. ☐

Claim — Let $f_1(x), g_1(x), f_2(x), g_2(x)$ be polynomials with real coefficients. Then the following holds.

$$\begin{aligned}&(f_1(x)^2 + g_1(x)^2)(f_2(x)^2 + g_2(x)^2)\\ &= (f_1(x)f_2(x) - g_1(x)g_2(x))^2 + (f_1(x)g_2(x) - f_2(x)g_1(x))^2\end{aligned}$$

Proof of the Claim. Note that

$$\begin{aligned}&(f_1(x)^2 + g_1(x)^2)(f_2(x)^2 + g_2(x)^2)\\ &= f_1(x)^2 f_2(x)^2 + g_1(x)^2 g_2(x)^2 - 2f_1(x)f_2(x)g_1(x)g_2(x)\\ &\quad + f_1(x)^2 g_2(x)^2 + f_2(x)^2 g_1(x)^2 + 2f_1(x)g_2(x)f_2(x)g_1(x)\\ &= (f_1(x)f_2(x) - g_1(x)g_2(x))^2 + (f_1(x)g_2(x) + f_2(x)g_1(x))^2.\end{aligned}$$

□

Combining the above Claims, and using induction, the result follows. ■

Example 1.24 (India RMO 2017a P3). Let $P(x) = x^2 + \frac{x}{2} + b$ and $Q(x) = x^2 + cx + d$ be two polynomials with real coefficients such that $P(x)Q(x) = Q(P(x))$ for all real x. Find all real roots of $P(Q(x)) = 0$.

Solution 18. Let α be a root of $P(x)$ in $\mathbb{C}$. Substituting $x = \alpha$ in $P(x)Q(x) = Q(P(x))$, we obtain $Q(0) = 0$, which shows that $d = 0$. This gives

$$P(x)(x^2 + cx) = P(x)(P(x) + c),$$

which yields $P(x) + c = x^2 + cx$. It follows that $c = \frac{1}{2}, b = -\frac{1}{2}$.

Note that the roots of $P(x)$ are $-1, \frac{1}{2}$. Hence, any root β of $P(Q(x)) = 0$ satisfies $Q(\beta) = -1$ or $Q(\beta) = \frac{1}{2}$. Since the discrimint of $Q(x) + 1$ is negative, it follows that if β is a real root of $P(Q(x)) = 0$, then $Q(\beta) = \frac{1}{2}$, which gives $\beta = -1$ or $\beta = 1/2$.

Note that $P(\frac{1}{2}) = 0$ and $-1, \frac{1}{2}$ are the roots of $Q(x) = 1/2$. This shows that the real roots of $P(Q(x)) = 0$ are $-1, \frac{1}{2}$. ■

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