# Invariance principle

MOPSS

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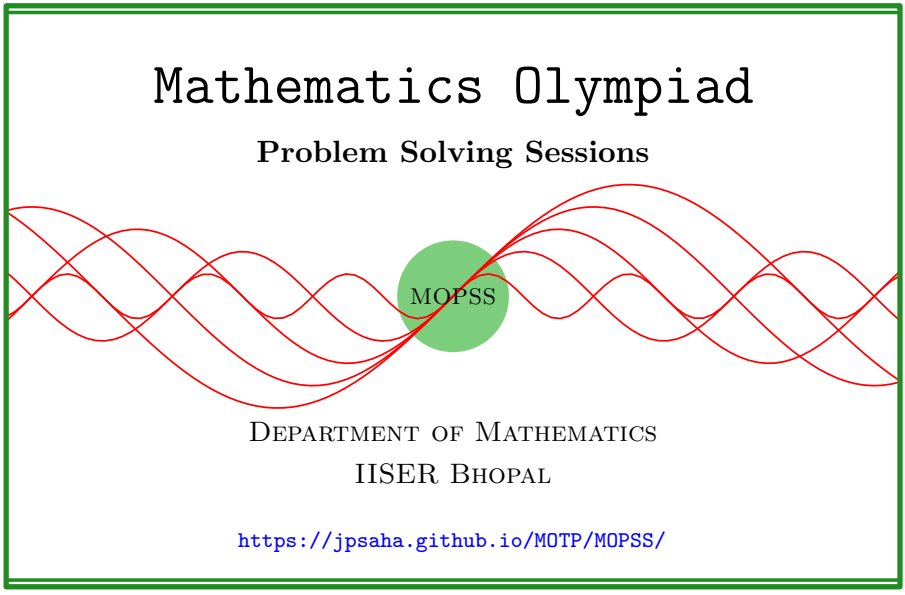


## Suggested readings

- Evan Chen's advice On reading solutions, available at https://blog.evanchen.cc/2017/03/06/on-reading-solutions/.
- Evan Chen's Advice for writing proofs/Remarks on English, available at https://web.evanchen.cc/handouts/english/english.pdf.
- Notes on proofs by Evan Chen from OTIS Excerpts [**Che25**, Chapter 1].
- Tips for writing up solutions by Edward Barbeau, available at https://www.math.utoronto.ca/barbeau/writingup.pdf.
- Evan Chen discusses why math olympiads are a valuable experience for high schoolers in the post on Lessons from math olympiads, available at https://blog.evanchen.cc/2018/01/05/lessons-from-math-olympiads/.

# List of problems and examples



# §1 Invariance principle

**Example 1.1** (Moscow MO 1959 Day 2 Grade 7 P5)**.** Consider $n$ numbers $x_1, \ldots, x_n$, each equal to 1 or $-1$. Prove that if

$$x_1x_2 + x_2x_3 + \cdots + x_{n-1}x_n + x_nx_1 = 0, \tag{1}$$

then $n$ is divisible by 4.

**Solution 1.** Since each of the summands of $x_1x_2 + x_2x_3 + \cdots + x_{n-1}x_n + x_nx_1$ is equal to 1 or $-1$, using Eq. (1), it follows that the number of the summands of $x_1x_2 + x_2x_3 + \cdots + x_{n-1}x_n + x_nx_1$ equal to 1 coincides with the number of the summands of $x_1x_2 + x_2x_3 + \cdots + x_{n-1}x_n + x_nx_1$ equal to $-1$. These two numbers add up to $n$, and hence $n$ is even, i.e. $n = 2m$ for some positive integer $m$.

Note that $m$ is equal to the number of the summands of $x_1x_2 + x_2x_3 + \cdots + x_{n-1}x_n + x_nx_1$, which are equal to $-1$. In other words, $m$ is equal to the number of *alterations*, i.e. pairs of consecutive terms of the sequence

$$x_1, x_2, \ldots, x_n, x_1,$$

whose signs differ. Since the initial and the final term of the above sequence are equal, their signs remain unchanged, and hence the number of the alterations of the above sequence has to be an even number. Since $m$ is even, it follows that $n$ is divisible by 4. ■

**Remark.** An elementary congruence argument may also be used to observe that $n$ is even. Note that

$$\begin{aligned} n &= x_1^2x_2^2 + x_2^2x_3^2 + \cdots + x_{n-1}^2x_n^2 + x_n^2x_1^2 \\ &\equiv x_1x_2 + x_2x_3 + \cdots + x_{n-1}x_n + x_nx_1 \pmod 2, \end{aligned}$$

and hence $n$ is even.

One may also proceed as follows. For $1 \le i \le n$, write

$$\varepsilon_i := \begin{cases} 1 & \text{if } x_i \text{ and } x_{i+1} \text{ are of the same sign,} \\ -1 & \text{otherwise,} \end{cases}$$

where $x_{n+1} := x_1$. In other words,

$$\varepsilon_i = x_ix_{i+1}.$$

Note that

$$\begin{aligned} n &= x_1^2x_2^2 + x_2^2x_3^2 + \cdots + x_{n-1}^2x_n^2 + x_n^2x_1^2 \\ &= \varepsilon_1x_1x_2 + \varepsilon_2x_2x_3 + \cdots + \varepsilon_{n-1}x_{n-1}x_n + \varepsilon_nx_nx_1 \\ &= (\varepsilon_1 - 1)x_1x_2 + (\varepsilon_2 - 1)x_2x_3 + \cdots + (\varepsilon_{n-1} - 1)x_{n-1}x_n + (\varepsilon_n - 1)x_nx_1 \\ &= 2 \sum_{1\le i\le n, \varepsilon_i = -1} (-x_ix_{i+1}) \\ &= 2 \times \text{the number of alternations of the sequence } x_1, \ldots, x_n, x_1. \end{aligned}$$

Note that to arrive at the above, we did not use the fact that $n$ is even. Moreover, the prior argument tells us that any (finite) sequence of $\pm 1$'s, with equal initial and final terms, have an even number of alterations. Hence, $n$ is a multiple of 4.

**Upshot** — At times, algebraic techniques might be quite useful.

**Example 1.2.** [Eng98, p. 5] Let $a_1, a_2, \ldots, a_n$ be $n$ numbers such that each $a_i$ is either 1 or $-1$. If

$$a_1a_2a_3a_4 + a_2a_3a_4a_5 + \cdots + a_na_1a_2a_3 = 0,$$

then prove that 4 divides $n$.

**Solution 2.** The solution relies on the following claim.

**Claim** — Let $y_1, y_2, \ldots, y_n$ be a sequence of integers such that each $y_i$ is 1 or $-1$. Let $1 \le m \le n$ be an integer. Then the sum

$$y_1y_2y_3y_4 + y_2y_3y_4y_5 + \cdots + y_ny_1y_2y_3,$$

and the new sum obtained by replacing $y_m$ by $-y_m$ in the above sum, leave the same remainder upon division[a] by 4.

[a] If $b$ is a positive integer, and $a$ is an integer, then there are integers $q, r$ such that

$$a = qb + r, \quad 0 \le r \le b$$

hold. The integer $q$ is called the *quotient* and $r$ is called the *remainder*.

Note that one may divide any integer (even the negative ones) by a positive integer and obtain the quotient and the remainder. For example, if 9 is divided by 4, it leaves a remainder of 1. If $-11$ is divided by 4, then (after writing $-11 = (-3)\cdot 4 + 1$, one finds that) it leaves a remainder of 1. If $-53$ is divided by 4, then (after writing $-53 = (-14)\cdot 4 + 3$, one finds that) it leaves a remainder of 3.

*Proof of the Claim.* In the above cyclic sum, $y_m$ appears in precisely four summands. Since the remaining summands are unchanged, it suffices to show that the sum of those four summands (denoted by $\mathcal{A}$), and the new sum obtained by replacing $y_m$ by $-y_m$ in $\mathcal{A}$ (to be denoted by $\mathcal{B}$), leave the same remainder upon division by 4.

In fact, it reduces to showing that if $\mathscr{A}$ is a sum of four terms where each term is equal to 1 or $-1$, then $\mathscr{A}$ and $-\mathscr{A}$ leave the same remainder upon division by 4, or equivalently, their difference, which is $2\mathscr{A}$ is a multiple of 4, which is equivalent to saying $\mathscr{A}$ is even.

If all the terms of $\mathscr{A}$ are equal, then $\mathscr{A}$ is even. If three of the terms of $\mathscr{A}$ are 1 and the remaining one is $-1$, then $\mathscr{A}$ is even. Moreover, if two of the terms of $\mathscr{A}$ are 1 and the other twos are $-1$, then $\mathscr{A}$ is even. Further, if only one term of $\mathscr{A}$ is 1 and the others are equal to $-1$, then $\mathscr{A}$ is also even. The Claim follows. □

In the given sum

$$a_1a_2a_3a_4 + a_2a_3a_4a_5 + \cdots + a_na_1a_2a_3,$$

we continue to replace all the $a_i$'s which are equal to $-1$ by 1's. In the first step, $S$ is divisible by 4. By the above Claim, at the end of each step, the new sum obtained is also divisible by 4. At the very last step, the sum will change to $n$, and hence, $n$ is divisible by 4. ■

For more exercises on the invariance principle, we refer to [**Eng98**, Chapter 1].

## References

[**Che25**] Evan Chen. *The OTIS Excerpts*. Available at https://web.evanchen.cc/excerpts.html. 2025, pp. vi+289

[**Eng98**] Arthur Engel. *Problem-solving strategies*. Problem Books in Mathematics. Springer-Verlag, New York, 1998, pp. x+403. ISBN: 0-387-98219-1 (cited pp. 3, 4)