# Integer divisibility

MOPSS

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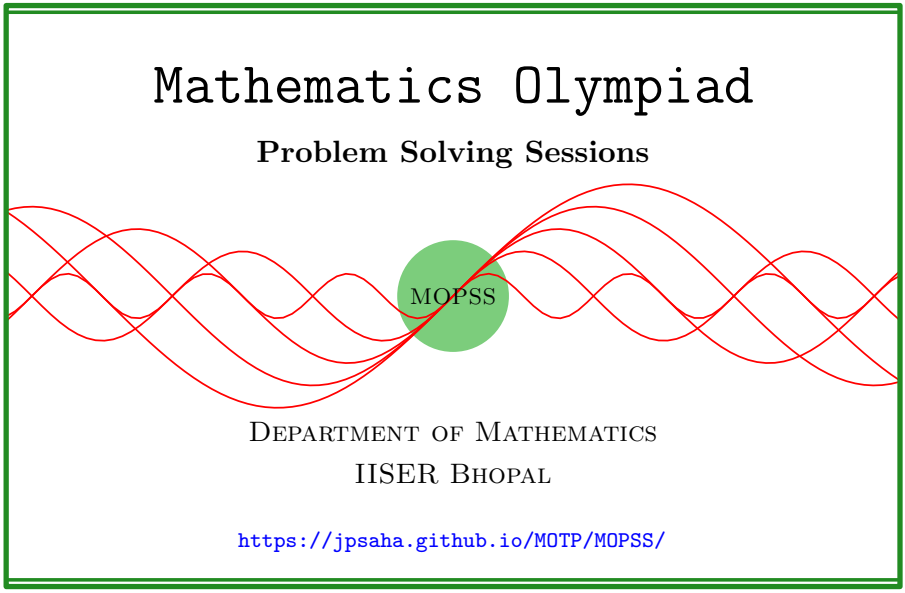


## Suggested readings

- Evan Chen's advice On reading solutions, available at https://blog.evanchen.cc/2017/03/06/on-reading-solutions/.
- Evan Chen's Advice for writing proofs/Remarks on English, available at https://web.evanchen.cc/handouts/english/english.pdf.
- Notes on proofs by Evan Chen from OTIS Excerpts [**Che25**, Chapter 1].
- Tips for writing up solutions by Edward Barbeau, available at https://www.math.utoronto.ca/barbeau/writingup.pdf.
- Evan Chen discusses why math olympiads are a valuable experience for high schoolers in the post on Lessons from math olympiads, available at https://blog.evanchen.cc/2018/01/05/lessons-from-math-olympiads/.

## List of problems and examples



# §1 Integer divisibility

**Lemma 1**

If $P$ is a polynomial with integer coefficients and $a, b$ are integers, then $P(a) - P(b)$ is a multiple of $a - b$.

**Example 1.1.** Let $P(x)$ be a polynomial with integer coefficients such that $P(0), P(1)$ are odd. Show that $P(x)$ does not have any integer root.

**Solution 1.** If $P(x)$ admits an integer root $\alpha$, then $\alpha, \alpha - 1$ are odd, which is impossible. ■

**Example 1.2** (India RMO 2016g P8)**.** At some integer points a polynomial with integer coefficients take values $1, 2$ and $3$. Prove that there exist not more than one integer at which the polynomial is equal to $5$.

**Solution 2.** Denote the polynomial by $P(x)$. On the contrary, let us assume that there are at least two distinct integers where $P(x)$ takes the value $5$.

Let $a, b, c$ be integers such that

$$P(a) = 1, P(b) = 2, P(c) = 3.$$

Note that $a - b$ divides $P(a) - P(b)$, $b - c$ divides $P(b) - P(c)$. It follows that $a - b = \pm 1, b - c = \pm 1$. Since $a, b$ are of opposite parity, and so are the integers $b, c$, we obtain that $a, c$ are of the same parity. Noting that $c - a$ divides $P(c) - P(a) = 2$, it follows that $c - a = \pm 2$. Combining this with $a - b = \pm 1, b - c = \pm 1$, we get $a - b = b - c = 1$ or $a - b = b - c = -1$.

This shows that $P(b - 1) = 1, P(b) = 2, P(b + 1) = 3$ holds or $P(b + 1) = 1, P(b) = 2, P(b - 1) = 3$ holds. Note that in the first case, the polynomial $R(x) := P(x - b)$ takes the values $1, 2, 3$ at the integers $-1, 0, 1$ respectively. In the second case, the polynomial $S(x) = P(-x + b)$ takes the values $1, 2, 3$ at

the integers $-1, 0, 1$ respectively. This proves that there is a polynomial $Q(x)$ with integer coefficients which takes the values $1, 2, 3$ at $-1, 0, 1$ respectively.

From the hypothesis, it follows that there are distinct integers $i, j$ such that $Q(i) = Q(j) = 5$. Note that $i - 1$ divides $Q(i) - Q(1) = 2$, $i$ divides $Q(i) - Q(0) = 3$, $i + 1$ divides $Q(i) - Q(-1) = 4$. Since $i$ divdes 3, we obtain $i = \pm 1, \pm 3$. Using $Q(-1) = 1, Q(1) = 3$, we get $i \neq -1, i \neq 1$. This gives $i = \pm 3$. Noting that $i - 1$ divides 2, we obtain $i \neq -3$, and hence $i = 3$. Similarly, it follows that $j = 3$. ■

**Example 1.3.** Let $P(x)$ be a polynomial with integer coefficients such that $P(20), P(25)$ are of absolute value equal to 1. Show that $P(x)$ does not vanish at any integer.

**Solution 3.** On the contrary, let us assume that $P(x)$ vanishes at an integer $\alpha$. Note that $\alpha - 20$ divides 1, and so does $\alpha - 25$. This shows that $\alpha - 20, \alpha - 25$ are absolute value equal to 1. Applying triangle inequality, we obtain

$$5 \leq |\alpha - 20| + |\alpha - 5| \leq 2,$$

which is impossible. ■

**Example 1.4** (USAMO 1974 P1)**.** Let $a$, $b$, and $c$ denote three distinct integers, and let $P$ denote a polynomial having all integral coefficients. Show that it is impossible that $P(a) = b$, $P(b) = c$, and $P(c) = a$.

**Solution 4.** Note that

$$a - b \mid P(a) - P(b) = b - c \mid P(b) - P(c) \mid c - a \mid P(c) - P(a) = a - b.$$

Consequently, the integers $a - b, b - c, c - a$ are of the same absolute value. Denote their absolute value by $k$. Note that their sum is zero. However, the sum is equal to $mk$, for some $m \in \{\pm 1, \pm 3\}$. Hence, $k$ is equal to zero.

This yields that $a = b = c$. ■

Here is a more general result.

**Example 1.5.** Let $P(x)$ be a polynomial with integer coefficients, and let $n$ be an odd positive integer. Suppose that $x_1, x_2, \ldots, x_n$ is a sequence of integers such that $x_2 = P(x_1), x_3 = P(x_2), \ldots, x_n = P(x_{n-1})$, and $x_1 = P(x_n)$. Prove that all the $x_i$'s are equal.

> **Walkthrough** — Show that
>
> $$a_1 - a_2 \mid a_2 - a_3 \mid a_3 - a_4 \mid \cdots \mid a_n - a_1 \mid a_1 - a_2.$$
>
> Note that sum of these differences is an odd multiple of their absolute value.

**Lemma 2**

Let $P$ be a polynomial with integer coefficients. Suppose $a$ is an integer and $k$ is a positive integer such that $P^k(a) = a$, where $P^k$ denotes the $k$-fold composite map from $\mathbb{Z} \to \mathbb{Z}$. Show that $P^2(a) = a$.

*Proof.* Let $\ell$ denote the smallest positive integer such that $P^\ell(a) = a$. If $\ell = 1$ or $\ell = 2$, then we are done. Henceforth, we assume that $\ell \geq 3$.

Note that

$$P(a) - a \mid P^2(a) - P(a) \mid \cdots \mid P^\ell(a) - P^{\ell-1}(a) = a - P^{\ell-1}(a) \mid P(a) - a.$$

Since $a - P^{\ell-1}(a)$ is nonzero, it follows that the above differences are nonzero. Consequently, for any $1 \leq i \leq \ell$,

$$P^{i+1}(a) - P^i(a) = \pm(P^i(a) - P^{i-1}(a)).$$

If $P^{i+1}(a) = P^{i-1}(a)$ holds for some $1 \leq i \leq \ell$, then applying $P^{\ell-i+1}$ to both sides, we obtain $P^2(a) = a$, which contradicts the assumption that $\ell \geq 3$. It follows that for any $1 \leq i \leq \ell$,

$$P^{i+1}(a) - P^i(a) = P(a) - a$$

holds, which implies that

$$\sum_{i=0}^{\ell-1} (P^{i+1}(a) - P^i(a)) = \ell(P(a) - a).$$

This gives $P(a) = a$, which contradicts the assumption that $\ell \geq 3$. This completes the proof. □

**Example 1.6** (IMO 2006 P5)**.** (Dan Schwarz, Romania) Let $P(x)$ be a polynomial of degree $n > 1$ with integer coefficients, and let $k$ be a positive integer. Consider the polynomial $Q(x) = P(P(\ldots P(P(x))\ldots))$, where $P$ occurs $k$ times. Prove that there are at most $n$ integers $t$ such that $Q(t) = t$.

**Solution 5.** By the above lemma, it reduces to considering the case $Q(x) = P^2(x)$.

Suppose $Q$ has more than $n$ fixed points. Since $P$ is not linear, it follows that $P$ cannot have $n$ fixed points, and hence not all the fixed points of $Q$ are fixed points of $P$. Let $b$ be a non-fixed point of $P$, and $Q(b) = b$. Suppose $a$ be a fixed point of $Q$, other than $b$.

Let us first consider the case that $P(a) \neq a$. Note that

$$P(b) - a \mid P(a) - b \mid P(b) - a$$

holds, and

$$a-b \mid P(a)-P(b) \mid a-b$$

holds too. This yields that

$$|P(a)-b| = |P(b)-a|, \quad |P(a)-P(b)| = |a-b|.$$

If

$$P(a)-b = a-P(b), \text{ and } P(a)-P(b) = a-b$$

hold, then $b$ would be a fixed point of $P$. It follows that at least one of

$$P(a)-b = -(a-P(b)), P(a)-P(b) = -(a-b)$$

holds. Consequently, we obtain

$$P(a)+a = P(b)+b.$$

Next, let us consider the case that $P(a) = a$. Note that

$$P(b)-a \mid b-a \mid P(b)-a.$$

Since $b$ is not a fixed point for $P$, it follows that

$$P(b)-a = a-b,$$

which yields

$$P(a)+a = P(b)+b.$$

This proves that all the roots of $Q(x) = x$ are the roots of $P(x)+x = P(b)+b$. Since $P(x)$ has degree $n > 1$, it follows that the polynomial $P(x)+x-P(b)-b$ is of degree $n$, and it has more than $n$ roots, which is impossible.

Hence, there are at most $n$ integers $t$ such that $Q(t) = t$ holds. ■

**Example 1.7** (Tournament of Towns, Spring 2014, Senior, A Level, P4 by G.K. Zhukov)**.** In the plane, the points with integer coordinates $(x, y)$ satisfying $0 \leq y \leq 10$ are marked. Consider a polynomial of degree 20 with integer coefficients. Determine the maximum possible number of marked points which can lie on its graph.

**Solution 6.** Note that the polynomial

$$(x-1)(x-2)(x-3)\ldots(x-20)$$

of degree 20 has integer roots. Let us prove that the graph of no polynomial of degree 20 with integer coefficients passes through more than 20 marked points.

**Claim —** Let $P(x)$ be a polynomial of degree 20 with integer coefficients. No more than 20 marked points lie on the graph of $P(x)$.

*Proof of the Claim.* On the contrary, let us assume that there are integers $x_1 < x_2 < \cdots < x_{21}$ such that

$$0 \leq P(x_i) \leq 10$$

holds for all $1 \leq i \leq 21$. For any integer $1 \leq i \leq 10$, the inequality $x_{21} - x_i \geq 11$ holds, and using that $x_{21} - x_i$ divides the integer $P(x_{21}) - P(x_i)$, which lies in $[-10, 10]$, it follows that $P(x_{21}) = P(x_i)$. Similarly, for any integer $12 \leq i \leq 21$, it follows that $P(x_1) = P(x_i)$. This shows that $P(x_i) = P(x_1)$ for any integer $i \in \{1, 2, \ldots, 10\} \cup \{12, 13, \ldots, 21\}$. Since $P(x)$ is a polynomial of degree 20, it follows that

$$P(x) - P(x_1) = c(x - x_1)(x - x_2)\ldots(x - x_{10})(x - x_{12})(x - x_{13})\ldots(x - x_{21})$$

holds for some nonzero integer $a$. This yields that

$$|P(x_{11}) - P(x_1)| \geq (10!)^2,$$

which is impossible. □

This proves that the maximum possible number of marked points which can lie on the graph of a polynomial of degree 20 with integer coefficients is equal to 20. ■

## References

[**Che25**] Evan Chen. *The OTIS Excerpts*. Available at https://web.evanchen.cc/excerpts.html. 2025, pp. vi+289 (cited p. 1)