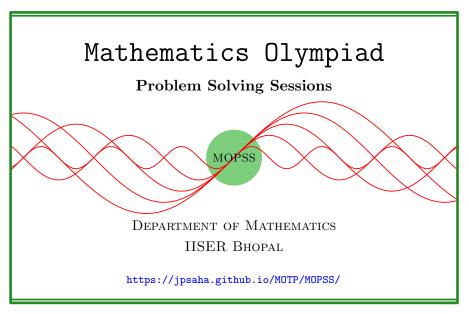
Functional equations

MOPSS

29 April 2025



Suggested readings

- Evan Chen's advice On reading solutions, available at https://blog.evanchen.cc/2017/03/06/on-reading-solutions/.
- Evan Chen's Advice for writing proofs/Remarks on English, available at https://web.evanchen.cc/handouts/english/english.pdf.
- Notes on proofs by Evan Chen from OTIS Excerpts [Che25, Chapter 1].
- Tips for writing up solutions by Edward Barbeau, available at https://www.math.utoronto.ca/barbeau/writingup.pdf.
- Evan Chen discusses why math olympiads are a valuable experience for high schoolers in the post on Lessons from math olympiads, available at https://blog.evanchen.cc/2018/01/05/lessons-from-math-olympiads/.

List of problems and examples

1.1	Example (cf. Canadian Mathematical Olympiad 1969 P8,	
	Australian Mathematics Competition 1984, India Pre-RMO	
	2012 P16)	2
1.2	Example (India RMO 2006 P7)	3

§1 Functional equations

Example 1.1 (cf. Canadian Mathematical Olympiad 1969 P8, Australian Mathematics Competition 1984, India Pre-RMO 2012 P16). [Tao06, Problem 3.2] Suppose

$$x_1, x_2, x_3, x_4, \dots$$

is a sequence of integers satisfying the following properties:

- $(1) x_2 = 2,$
- (2) $x_{mn} = x_m x_n$ for all positive integers m, n,
- (3) $x_m < x_n$ for any positive integers m, n with m < n.

Find x_{2024} . What is the value of the sum

$$x_1 + x_2 + \dots + x_{2024}$$
?

Summary — Observe that $x_{2^n} = 2^n$ for any $n \ge 1$. Combining this with the hypothesis that $\{x_n\}_{n\ge 1}$ is an increasing sequence of **positive integers**, conclude that $x_n = n$ for any $n \ge 1$.

Walkthrough —

- (a) What can be said about x_4, x_8, x_{16}, x_{32} ?
- (b) Note that $x_4 = x_{2\times 2}, x_8 = x_{4\times 2}, x_{16} = x_{8\times 2}, x_{32} = x_{16\times 2}$.
- (c) Can one show that $x_{2^n} = 2^n$ for any $n \ge 1$?
- (d) Show that $x_m = m$ for any $m \ge 1$ (does property (3) help?).

Solution 1. From the second condition, we obtain

$$x_{2^n} = x_2^n$$

for any integer $n \geq 1$. Using the first condition, it gives

$$x_{2n} = 2^n$$

for any integer $n \ge 1$. Since $\{x_n\}_{n\ge 1}$ is an increasing sequence of positive integers, it follows that $x_n = n$ for any positive integer n. This gives

$$x_{2024} = 2024.$$

It follows that

$$x_1 + x_2 + \dots + x_{2024} = 1012 \cdot 2025 = 2049300.$$

Example 1.2 (India RMO 2006 P7). Let X be the set of all positive integers greater than or equal to 8, and let $f: X \to X$ be a function such that f(x+y) = f(xy) for all $x \ge 4$, $y \ge 4$. If f(8) = 9, determine f(9).

Walkthrough — Using the given condition, try to express f(9) in terms of f(8).

Solution 2. Note that

$$f(9) = f(4+5) = f(20) = f(4+16) = f(64),$$

and

$$f(8) = f(4+4) = f(16) = f(8+8) = f(64).$$

Using the given condition f(8) = 9, it follows that f(9) = 9.

References

- [Che25] EVAN CHEN. The OTIS Excerpts. Available at https://web.evanchen.cc/excerpts.html. 2025, pp. vi+289 (cited p. 1)
- [Tao06] Terence Tao. Solving mathematical problems. A personal perspective. Oxford University Press, Oxford, 2006, pp. xii+103. ISBN: 978-0-19-920560-8; 0-19-920560-4 (cited p. 2)