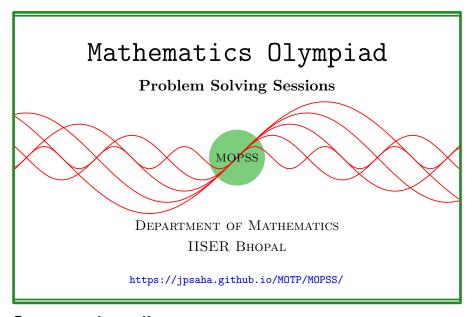
Differentiation and multiple roots

MOPSS

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Suggested readings

- Evan Chen's advice On reading solutions, available at https://blog.evanchen.cc/2017/03/06/on-reading-solutions/.
- Evan Chen's Advice for writing proofs/Remarks on English, available at https://web.evanchen.cc/handouts/english/english.pdf.
- Notes on proofs by Evan Chen from OTIS Excerpts [Che25, Chapter 1].
- Tips for writing up solutions by Edward Barbeau, available at https://www.math.utoronto.ca/barbeau/writingup.pdf.
- Evan Chen discusses why math olympiads are a valuable experience for high schoolers in the post on Lessons from math olympiads, available at https://blog.evanchen.cc/2018/01/05/lessons-from-math-olympiads/.

List of problems and examples

1.1 Example (Putnam 1956 B7, IMOSL 1981 Cuba) 2

§1 Differentiation and multiple roots

Lemma 1

Let P(x) be a polynomial with complex coefficients, and α be a complex number. Then α is a root of P(x) having multiplicity at least $r \geq 2$ (i.e., $(x-\alpha)^r$ divides P(x)) if and only if it is a root of $P(x), P'(x), \ldots, P^{(r)}(x)$, where $P^{(r)}(x)$ denotes the r-fold derivative of P(x).

Example 1.1 (Putnam 1956 B7, IMOSL 1981 Cuba). The polynomials P(z) and Q(z) with complex coefficients have the same set of numbers for their zeroes but possibly different multiplicities. The same is true of the polynomials P(z)+1 and Q(z)+1. **Assume that at least one of** P(z), Q(z) is nonconstant. Prove that P(z)=Q(z).

Walkthrough —

- (a) Assume that $\deg P \ge \deg Q$.
- (b) Denote these two set of roots by S_1, S_2 . Considering multiplicities, show that

$$2 \deg P - |S_1| - |S_2| \le \deg P' = \deg P - 1,$$

which yields

$$|S_1| + |S_2| > \deg P.$$

(c) Note that P-Q vanishes at the elements of $S_1 \cup S_2$, which has size larger than the degree of P-Q.

Solution 1. On the contrary, let us assume that $P \neq Q$. Without loss of generality, let us assume that $\deg P \geq \deg Q$. Let S_1 (resp. S_2) denote the common set of zeroes of P, Q (resp. P+1, Q+1). For a polynomial f(x), let us denote its multiset of zeroes by $\mathcal{Z}(f)$.

Note that $\mathcal{Z}(P')$ contains $\mathcal{Z}(P) \setminus S_1$, and $\mathcal{Z}(P')$ also contains $\mathcal{Z}(P+1) \setminus S_2$. Since $\mathcal{Z}(P)$ and $\mathcal{Z}(P+1)$ are disjoint, it follows that

$$2 \deg P - |S_1| - |S_2| \le \deg P' < \deg P - 1,$$

where the final step holds since $\deg P \ge \deg Q$, and one of P,Q is nonconstant. This gives that $|S_1| + |S_2| > \deg P$.

Note that S_1, S_2 are disjoint, and

$$P(z) - Q(z) = (P(z) + 1) - (Q(z) + 1)$$

holds. It follows that P-Q vanishes at $S_1 \cup S_2$, and hence $\deg P \ge |S_1| + |S_2|$. This contradicts the inequality $|S_1| + |S_2| > \deg P$. Consequently, we obtain P = Q.

References

[Che25] EVAN CHEN. The OTIS Excerpts. Available at https://web.evanchen.cc/excerpts.html. 2025, pp. vi+289