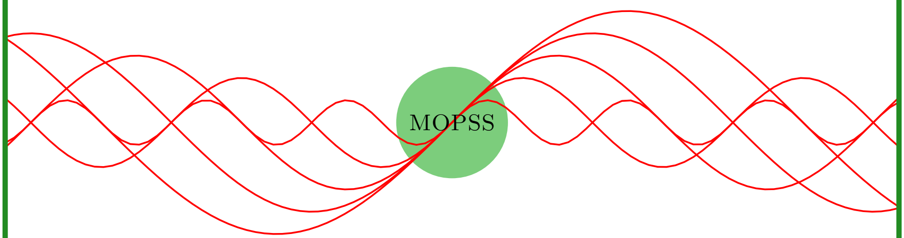
Binomial theorem

MOPSS

28 April 2025

Mathematics Olympiad

Problem Solving Sessions



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https://jpsaha.github.io/MOTP/MOPSS/

Suggested readings

- Evan Chen's advice On reading solutions, available at https://blog.evanchen.cc/2017/03/06/on-reading-solutions/.
- Evan Chen's Advice for writing proofs/Remarks on English, available at https://web.evanchen.cc/handouts/english/english.pdf.
- Notes on proofs by Evan Chen from OTIS Excerpts [**Che25**, Chapter 1].
- Tips for writing up solutions by Edward Barbeau, available at https://www.math.utoronto.ca/barbeau/writingup.pdf.
- Evan Chen discusses why math olympiads are a valuable experience for high schoolers in the post on Lessons from math olympiads, available at https://blog.evanchen.cc/2018/01/05/lessons-from-math-olympiads/.

List of problems and examples

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§1 Binomial theorem

Example 1.1 (India RMO 1996 P6)**.** Given any positive integer n, show that there are two positive rational numbers a and b, $a \neq b$, which are not integers and which are such that $a - b, a^2 - b^2, a^3 - b^3, \dots, a^n - b^n$ are all integers.

Walkthrough —

(a) We may expect that $a = c + \frac{1}{2}$ and $b = d + \frac{1}{2}$ would work where c, d are suitable integers. We need to see for what choices of c, d, the difference $a^m - b^m$ is an integer for all $1 \leq m \leq n$.

(b) By the binomial theorem [a],

$$a^m = c^m + \binom{m}{1} c^{m-1} \frac{1}{2} + \binom{m}{2} c^{m-2} \frac{1}{2^2} + \cdots + \binom{m}{m-1} c \frac{1}{2^{m-1}} + \frac{1}{2^m},$$
$$b^m = d^m + \binom{m}{1} d^{m-1} \frac{1}{2} + \binom{m}{2} d^{m-2} \frac{1}{2^2} + \cdots + \binom{m}{m-1} d \frac{1}{2^{m-1}} + \frac{1}{2^m}.$$

This shows

$$\begin{aligned} a^m - b^m = (c^m - d^m) &+ \binom{m}{1} \frac{c^{m-1} - d^{m-1}}{2} \\ &+ \binom{m}{2} \frac{c^{m-2} - d^{m-2}}{2^2} + \cdots + \binom{m}{m-1} \frac{c-d}{2^{m-1}}. \end{aligned}$$

(c) It follows that $a^m - b^m$ would be an integer if we could ensure that

$$\frac{c^{m-1} - d^{m-1}}{2}, \frac{c^{m-2} - d^{m-2}}{2^2}, \dots, \frac{c-d}{2^{m-1}}$$

are all integers.

(d) Note that $c^{m-1} - d^{m-1}, c^{m-2} - d^{m-2}, \dots, c - d$ are all divisible by $c - d$. So it would be enough to find suitable integers c, d such that $c - d$ is divisible by $2, 2^2, \dots, 2^{m-1}$ for all $1 \leq m \leq n$.

(e) For instance, if we take $c = 2^{n-1}, d = 0$, that is,

$$a = 2^{n-1} + \frac{1}{2}, b = \frac{1}{2},$$

then $a-b, a^2-b^2, a^3-b^3, \ldots, a^n-b^n$ are all integers and a, b are positive rationals and not integers.

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[a]The binomial theorem says that

$$(x+y)^n = x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n-2}x^2y^{n-2} + \binom{n}{n-1}xy^{n-1} + y^n,$$

where $\binom{n}{i}$ is a positive integer and is equal to $\frac{n!}{i!(n-i)!}$ for $1 \leq i \leq n-1$. Indeed, if we expand $(x+y)^n$, then it becomes immediately clear that there exist integers $c_1, \ldots, c_{n-1}$ such that

$$(x+y)^n = x^n + c_1x^{n-1}y + c_2x^{n-2}y^2 + \cdots + c_{n-2}x^2y^{n-2} + c_{n-1}xy^{n-1} + y^n$$

holds, and moreover, the integers $c_1, \ldots, c_{n-1}$ do not depend on x and y, they depend only on n. A careful inspection of the above equality shows that for any $1 \leq i \leq n$, the integer c_i is equal to the number of ways of selecting a set of i elements from a set of n elements, which is denoted by $\binom{n}{i}$ by convention. This essentially proves (combined with the details to be filled in) the binomial theorem, which states that for any positive integer n,

$$(x+y)^n = x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n-2}x^2y^{n-2} + \binom{n}{n-1}xy^{n-1} + y^n$$

holds. However, it is not immediate that $\binom{n}{i}$ is equal to $\frac{n!}{i!(n-i)!}$. But we do not require it in this problem.

Solution 1. Take $a = 2^m + \frac{1}{2}$ and $b = \frac{1}{2}$, where m is a suitable positive integer to be determined later. Using the binomial theorem, we obtain

$$a^k = 2^{mk} + \binom{k}{1}\frac{2^{m(k-1)}}{2} + \binom{k}{2}\frac{2^{m(k-2)}}{2^2} + \cdots + \binom{k}{k-1}\frac{2^m}{2^{k-1}} + \frac{1}{2^k}$$

for any integer $k \geq 1$. We would like to have that $a-b, a^2-b^2, a^3-b^3, \ldots, a^n-b^n$ are all integers. Note that it suffices to make sure that $m \geq k-1$ for $k = 1, 2, \ldots, n$. Let us take

$$a = 2^n + \frac{1}{2}, b = \frac{1}{2}.$$

Note that the positive rational numbers a, b are not integers, and $a-b, a^2-b^2, a^3-b^3, \ldots, a^n-b^n$ are all integers. ■

Example 1.2 (Putnam 2004 B2)**.** Let m and n be positive integers. Show that

$$\frac{(m+n)!}{(m+n)^{m+n}} < \frac{m!n!}{m^m n^n}.$$

Solution 2. Applying the binomial theorem, we observe that $\binom{m+n}{m}m^m n^n$ is one of the terms of the binomial expansion of $(m+n)^{m+n}$. Noting that $\binom{m+n}{m}m^m n^n = \frac{(m+n)!}{m!n!}m^m n^n$, the result follows. ■

References

[**Che25**] Evan Chen. *The OTIS Excerpts*. Available at https://web.evanchen.cc/excerpts.html. 2025, pp. vi+289