Arithmetic progressions

MOPSS

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Mathematics Olympiad

Problem Solving Sessions

MOPSS

Department of Mathematics
IISER Bhopal

https://jpsaha.github.io/MOTP/MOPSS/

Suggested readings

- Evan Chen's advice On reading solutions, available at https://blog.evanchen.cc/2017/03/06/on-reading-solutions/.
- Evan Chen's Advice for writing proofs/Remarks on English, available at https://web.evanchen.cc/handouts/english/english.pdf.
- Notes on proofs by Evan Chen from OTIS Excerpts [**Che25**, Chapter 1].
- Tips for writing up solutions by Edward Barbeau, available at https://www.math.utoronto.ca/barbeau/writingup.pdf.
- Evan Chen discusses why math olympiads are a valuable experience for high schoolers in the post on Lessons from math olympiads, available at https://blog.evanchen.cc/2018/01/05/lessons-from-math-olympiads/.

List of problems and examples

§1 AP, GP, HP

Example 1.1. Show that there is a coloring of the positive integers using two colors such that there is no monochromatic infinite arithmetic progression.

We give a solution from this page.

Solution 1. Colour the first one red, the next two blue, the next three red, the next four blue and so on.

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 ...

Given an infinite arithmetic progression of positive integers with common difference d, note that any set of consecutive d positive integers contains a term of that progression. Considering a set of at least d consecutive red integers, and a set of at least d consecutive blue integers, it follows that there is no monochromatic infinite arithmetic progression. ■

Example 1.2. [PK74, Problem 47.3] Determine m so that the equation

$$x^4 - (3m+2)x^2 + m^2 = 0$$

has four real roots in arithmetic progression.

Solution 2. Let m be such that the given equation has four real roots in arithmetic progression. Denote the roots of this equation by

$$a - 3d, a - d, a + d, a + 3d.$$

Note that the sum of the roots is equal to 0, which gives $a = 0$. Also note that

$$-3d(-d + d + 3d) - d^2 - 3d^2 + 3d^2 = -(3m+2),$$

which yields $10d^2 = 3m + 2$. Moreover, we also have

$$9d^4 = m^2,$$

which yields $3d^2 = \pm m$. Combining this with $10d^2 = 3m + 2$, we obtain

$$m = \begin{cases} 6 & \text{if } d^2 = \frac{m}{3}, \\ -\frac{6}{19} & \text{if } d^2 = -\frac{m}{3}. \end{cases}$$

Let us determine whether for $m = 6$ and $m = -\frac{6}{19}$, the given equation has four real roots in arithmetic progression. Note that if $m = 6$, then

$$\begin{aligned} x^4 - (3m+2)x^2 + m^2 &= x^4 - 20x^2 + 36 \\ &= (x^2 - 2)(x^2 - 18) \\ &= (x + 3\sqrt{2})(x + \sqrt{2})(x - \sqrt{2})(x - 3\sqrt{2}). \end{aligned}$$

Moreover, if $m = -\frac{6}{19}$, then

$$\begin{aligned} x^4 - (3m+2)x^2 + m^2 &= x^4 - \frac{20}{19}x^2 + \frac{6^2}{19^2} \\ &= \left(x^2 - \frac{2}{19}\right)\left(x^2 - \frac{18}{19}\right) \\ &= \left(x + \frac{3\sqrt{2}}{\sqrt{19}}\right)\left(x + \frac{\sqrt{2}}{\sqrt{19}}\right)\left(x - \frac{\sqrt{2}}{\sqrt{19}}\right)\left(x - \frac{3\sqrt{2}}{\sqrt{19}}\right). \end{aligned}$$

This proves that the required values for m are

$$6, -\frac{6}{19}.$$

■

Example 1.3. [Kos14, Example 2.11] Let x, y be positive integers satisfying the Pell's equation $x^2 - 2y^2 = -1$. Prove that

$$1^3 + 3^3 + 5^3 + \cdots + (2y-1)^3 = x^2y^2.$$

Solution 3. Note that

$$\begin{aligned} &1^3 + 3^3 + 5^3 + \cdots + (2y-1)^3 \\ &= 1^3 + 2^3 + 3^3 + \cdots + (2y)^3 - \left(2^3 + 4^3 + 6^3 + \cdots + (2y)^3\right) \\ &= (y(2y+1))^2 - 8\left(1^3 + 2^3 + 3^3 + \cdots + y^3\right) \\ &= (y(2y+1))^2 - 2\left(y(y+1)\right)^2 \\ &= y^2(4y^2 + 4y + 1 - 2y^2 - 4y - 2) \end{aligned}$$

$$
\begin{aligned}
&= y^2(2y^2 - 1) \\
&= x^2y^2.
\end{aligned}
$$

■

Example 1.4 (India RMO 1994 P1)**.** A leaf is torn from a paperback novel. The sum of the numbers on the remaining pages is 15000. What are the page numbers on the torn leaf?

Solution 4. Suppose $1, 2, \ldots, n$ denote the page numbers of the novel and $x, x+1$ denote the page numbers of the torn leaf. The given condition implies that

$$
\frac{1}{2}n(n+1) = 15000 + 2x + 1. \tag{1}
$$

Since $1 \leq x \leq n-1$, we get

$$
15000 + 3 \leq \frac{1}{2}n(n+1) \leq 15000 + 2n - 1,
$$

which shows that

$$
n^2 + n - (30000 + 6) \geq 0, \quad n^2 - 3n - (30000 - 2) \leq 0.
$$

Thus

$$
\frac{-1 + \sqrt{120000 + 25}}{2} \leq n \leq \frac{3 + \sqrt{120000 + 1}}{2}.
$$

Now it can be checked that [1]

$$
\frac{3 + \sqrt{120000 + 1}}{2} < 175, \quad \frac{-1 + \sqrt{120000 + 25}}{2} > 172.
$$

Consequently, n is equal to 173 or 174. Since the number of pages of a novel is an even number, we conclude that $n = 174$. This shows that

$$
\begin{aligned}
2x &= 87 \times 175 - 15001 \\
&= 80 \times 175 + 50 \times 25 - 15026,
\end{aligned}
$$

and hence

$$
\begin{aligned}
x &= 40 \times 175 + 25 \times 25 - 7513 \\
&= 7000 + 625 - 7513 \\
&= 112.
\end{aligned}
$$

Consequently, the page numbers on the torn leaf are $112, 113$[2]. ■

[1]Since n is close to $\frac{1}{2}\sqrt{120000} = 100\sqrt{3}$, which is close 173. So we tried to bound n using integers close to 173, whose squares can be easily computed since squaring 175 is easy, at least once it is multiplied by 2.

[2]Do you find something wrong with it? One may note that substituting $n = 173$ in Equation (1) would yield $x = 25$, implying that the page numbers on the torn leaf are $25, 26$.

Example 1.5 (India RMO 2009 P6)**.** In a book with page numbers from 1 to 100 some pages are torn off. The sum of the numbers on the remaining pages is 4949. How many pages are torn off?

Solution 5. Suppose r pages are torn off. Denote the page numbers of the torn pages by $2n_1 - 1, 2n_1, 2n_2 - 1, 2n_2, \dots, 2n_r - 1, 2n_r$. So

$$\begin{aligned} &1 + 2 + \cdots + 50 \\ &= 4949 + 2n_1 - 1 + 2n_1 + 2n_2 - 1 + 2n_2 + \cdots + 2n_r - 1 + 2n_r, \end{aligned}$$

which gives

$$4(n_1 + \cdots + n_r) - r = 5050 - 4949 = 101. \tag{2}$$

Consequently, we obtain

$$\begin{aligned} 101 &= 4(n_1 + \cdots + n_r) - r \\ &\geq 4(1 + 2 + \cdots + r) - r \\ &= 2r(r+1) - r \\ &= r(2r+1), \end{aligned}$$

and hence

$$r \leq \frac{-1 + \sqrt{1 + 4 \cdot 2 \cdot 101}}{4} = \frac{-1 + \sqrt{809}}{4} \leq 7.$$

Moreover, Equation (2) implies that $r \equiv 3 \pmod 4$. Consequently, three pages are torn. ■

Example 1.6 (India RMO 2011b P3)**.** Let $a, b, c > 0$. If $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in arithmetic progression, and if a^2+b^2, b^2+c^2, c^2+a^2 are in geometric progression, show that $a = b = c$.

Solution 6. The given conditions yield

$$\frac{1}{b} - \frac{1}{a} = \frac{1}{c} - \frac{1}{b}, \quad \frac{b^2 + c^2}{a^2 + b^2} = \frac{c^2 + a^2}{b^2 + c^2}.$$

The first condition gives

$$\frac{a-b}{ab} = \frac{b-c}{bc} = \frac{a-c}{b(a+c)}. \tag{3}$$

On the contrary, let us assume that $a \neq c$. Using $a, c > 0$, we get $a^2 \neq c^2$, and hence, the second condition gives

$$\frac{b^2 + c^2}{a^2 + b^2} = \frac{c^2 + a^2}{b^2 + c^2} = \frac{a^2 - b^2}{c^2 - a^2} = \frac{a+b}{c+a}\frac{a-b}{c-a} = -\frac{a+b}{c+a}\frac{a}{a+c},$$

which is impossible since a, b, c are positive. This proves that $a = c$. Using Equation (3), it follows that $a = c$. This completes the proof. ■

Example 1.7 (India RMO 2014a P2)**.** Let $a_1, a_2, \ldots, a_{2n}$ be an arithmetic progression of positive real numbers with common difference d. Let

$$\sum_{i=1}^{n} a_{2i-1}^2 = x, \quad \sum_{i=1}^{n} a_{2i}^2 = y, \quad a_n + a_{n+1} = z.$$

Express d in terms of x, y, z, n.

Solution 7. Note that

$$\begin{aligned} y - x &= (a_2^2 - a_1^2) + (a_4^2 - a_3)^2 + \cdots + (a_{2n}^2 - a_{2n-1}^2) \\ &= d(a_1 + a_2 + a_3 + a_4 + \cdots + a_{2n-1} + a_{2n}). \end{aligned}$$

Note that $a_i + a_{2n-i}$ is equal to $a_n + a_{n+1}$ for any $0 < i < 2n$. Indeed,

$$\begin{aligned} a_i + a_{2n-i} &= a_1 + (i-1)d + a_1 + (2n-i-1)d \\ &= 2a_1 + (2n-2)d \end{aligned}$$

is independent of i. This shows that

$$y - x = dn(a_n + a_{n+1}) = dnz.$$

which is equal to dnz. Since $a_1, a_2, \ldots, a_{2n}$ is an arithmetic progression of positive real numbers, it follows that $z = a_n + a_{n+1}$ is nonzero, and consequetly,

$$d = \frac{y-x}{nz}.$$

■

Example 1.8 (India RMO 2016a P6)**.** Let $\langle a_1, a_2, a_3, \ldots \rangle$ be a strictly increasing sequence of positive integers in an arithmetic progression. Prove that there is an infinite subsequence of the given sequence whose terms are in a geometric progression.

> **Walkthrough** — For any positive integer d, note that
>
> $$1, 1+d, (1+d)^2, (1+d)^3, \ldots$$
>
> is a geometric progression, and it is a subsequence of the arithmetic progression with first term equal to 1, and having common difference d. Does this help?

Solution 8. For any positive integers a and d, note that

$$a, a(1+d), a(1+d)^2, \ldots$$

forms a geometric progression, and it is a subsequence of the arithmetic progression, with first term equal to a and common difference d. ■

Example 1.9 (India RMO 2016b P6)**.** Show that the infinite arithmetic progression $\langle 1, 4, 7, 10, \ldots \rangle$ has infinitely many 3-term subsequences in harmonic progression such that for any two such triples $\langle a_1, a_2, a_3 \rangle$ and $\langle b_1, b_2, b_3 \rangle$ in harmonic progression, one has

$$\frac{a_1}{b_1} \neq \frac{a_2}{b_2}.$$

Walkthrough —

(a) Try to find out a few 3-term harmonic progressions within the given sequence.

(b) Note that $1, 4$ cannot be extended to such a progression, since

$$\frac{1}{c} = \frac{2}{4} - \frac{1}{1}$$

has no solution in the positive integers.

(c) Note that $4, 7$ extends to the harmonic progression $4, 7, 28$ (solve the analogue of the above equation.).

(d) How to obtain more of such progressions? Let's have a look at $4, 7, 28$, and try to find **"why it works"**. Note that

$$\frac{2}{7} - \frac{1}{4} = \frac{2 \cdot 4 - 7}{4 \cdot 7} = \frac{1}{4 \cdot 7}.$$

This suggests to consider $n, 2n - 1, n(2n - 1)$, and check that it **does** form a harmonic progression.

(e) Does there exist enough positive integers n such that the terms of the harmonic progression $n, 2n - 1, n(2n - 1)$ are congruent to 1 modulo 3?

Solution 9. Note that if n is a positive integer and $n \equiv 1 \pmod 3$, then the integers $n, 2n - 1, n(2n - 1)$ are congruent to 1 modulo 3, and they form a harmonic progression. Making n vary over the positive integers of the form $3k + 1$, we obtain infinitely many 3-term harmonic progressions. Moreover, two such triples satisfy the desired inequality. ■

Example 1.10 (India RMO 2016c P6)**.**

(a) Given any natural number N, prove that there exists a strictly increasing sequence of N positive integers in harmonic progression.

(b) Prove that there cannot exist a strictly increasing infinite sequence of positive integers which is in harmonic progression.

Walkthrough — If a, c are positive integers, then note that

$$a(a+c), 2ac, c(a+c)$$

form a harmonic progression. Moreover, if a, c, e are three positive integers forming a harmonic progression, then it follows that there exist positive integers λ, μ, b, d such that $\lambda a, b, \lambda c$, and $\mu c, d, \mu e$ are harmonic progressions. Consequently,

$$\lambda \mu a, \mu b, \lambda \mu c, \lambda d, \lambda \mu e$$

is a harmonic progression.

Solution 10. To prove part (a), let us establish the following claim.

Claim — If a, c are positive integers, then the integers

$$a(a+c), 2ac, c(a+c)$$

form a harmonic progression. Consequently, given two positive integers, they can be scaled by a positive integers, such that the products form the first and the last term of a 3-term harmonic progression.

Proof of the Claim. The claim follows since

$$\frac{1}{a(a+c)} + \frac{1}{c(a+c)} = \frac{2}{2ac}$$

holds. □

Claim — Let $k \geq 3$ be a positive integer. For any positive integers $x_1, x_3, x_5, \ldots, x_{2k-1}$ in a harmonic progression, there exist positive integers $x_2, x_4, x_6, \ldots, x_{2k-2}$ and a positive integer λ such that the integers

$$\lambda x_1, x_2, \lambda x_3, x_4, \lambda x_5, x_6, \ldots, x_{2k-2}, \lambda x_{2k-1}$$

form a harmonic progression. Moreover, if $x_1, x_3, \ldots, x_{2k-1}$ is strictly increasing, then so is the above sequence.

Proof of the Claim. Let $x_1, x_3, x_5, \ldots, x_{2k-1}$ be positive integers in a harmonic progression. For any $2 \leq i \leq k$, there exist positive integers λ_i and x_{2i-2} such that the integers $\lambda_i x_{2i-3}, x_{2i-2}, \lambda_i x_{2i-1}$ are in a harmonic progression. Write $\lambda = \prod_{2 \leq i \leq k} \lambda_i$. It follows that for any $2 \leq i \leq k$, the integers $\lambda x_{2i-3}, \lambda x_{2i-2}/\lambda_i, \lambda x_{2i-1}$ are in a harmonic progression. Consequently, the integers

$$\lambda x_1, \lambda x_2/\lambda_2, \lambda x_3, \lambda x_4/\lambda_3, \lambda x_5, \lambda x_6/\lambda_4, \lambda x_7, \ldots, \lambda x_{2k-2}/\lambda_k, \lambda x_{2k-1}$$

form a harmonic progression. This proves the first part of the Claim. The second part follows. □

Since $3, 4, 6$ is a harmonic progression of strictly increasing positive integers, by the above Claim, it follows that there exist harmonic progressions of arbitrarily large length, formed by stricly increasing sequences of positive integers. This proves part (a).

To prove part (b), let us assume on the contrary that there exists a strictly increasing infinite sequence $a_1, a_2, a_3, \ldots$ of positive integers, which is in a harmonic progression. For any integer $k \geq 2$, note that a_1, a_k, a_{2k-1} is in harmonic progression, and it follows that

$$\begin{aligned} a_k &= 2\frac{a_1 a_{2k+1}}{a_1 + a_{2k+1}} \\ &= 2a_1 \left(1 - \frac{a_1}{a_1 + a_{2k+1}}\right) \\ &< 2a_1. \end{aligned}$$

This shows that the terms of the sequence $a_1, a_2, a_3, \ldots$ are bounded from the above. This is impossible since $a_1, a_2, a_3, \ldots$ is a strictly increasing sequence of positive integers. This completes the proof. ■

Example 1.11 (India RMO 2016d P6).

(i) Prove that if an infinite sequence of strictly increasing positive integers in arithmetic progression has one cube then it has infinitely many cubes.

(ii) Find, with justification, an infinite sequence of strictly increasing positive integers in arithmetic progression which does not have any cube.

Solution 11. Let us first establish part (i). Suppose we are given an infinite sequence of strictly increasing positive integers in arithmetic progression. Let a denote a term of this progression, and assume that $a = b^3$ for some integer b. Let d denote the common difference of the progression. Let k be a positive integer. Write $x = kd$. Note that

$$\begin{aligned} x^3 + 3x^2b + 3xb^2 &\geq \frac{1}{2}x^3 + 3x^2b + \frac{1}{2}x^3 + 3xb^2 \\ &\geq \frac{1}{2}x^3 + 3x^2b + \frac{1}{2}x^3 + 3xb^2 \\ &\geq \frac{1}{2}x^2(x + 6b) + \frac{1}{2}x(x^2 + b^2) \\ &> 0 \end{aligned}$$

holds if $x > 6\,|b|$. For any integer $k > 6\,|b|$, the integer $x^3 + 3x^2b + 3xb^2$ is a multiple of d, and the term $x^3 + 3x^2b + 3xb^2 + a$ of the given progression is equal to the perfect cube $(x+b)^3$. Consequently, the given progression has infinitely many cubes.

To prove part (ii), note that no even cube is congruent to 2 modulo 8. Hence, the arithmetic progression, having 2 as its first term, and with common difference 8, does not have any cube. ■

Example 1.12 (India RMO 2018b P2)**.** Find the set of all real values of a for which the real polynomial equation $P(x) = x^2 - 2ax + b = 0$ has real roots, given that $P(0) \cdot P(1) \cdot P(2) \neq 0$ and $P(0), P(1), P(2)$ form a geometric progression.

Solution 12. Let a, b be real numbers such that the polynomial $x^2 - 2ax + b$ has real roots, the product $P(0) \cdot P(1) \cdot P(2)$ is nonzero, and $P(0), P(1), P(2)$ form a geometric progression. Since $x^2 - 2ax + b$ has real roots, we get $a^2 \geq b$. Using $P(0), P(1), P(2)$ form a geometric progression, we obtain

$$b(4 - 4a + b) = (1 - 2a + b)^2,$$

which implies

$$4b - 4ab + b^2 = 1 + 4a^2 + b^2 - 4a - 4ab + 2b,$$

which gives $2b = (2a-1)^2$. Combining the above with $a^2 \geq b$ yields $2a^2 \geq (2a-1)^2$, which implies that $2a^2 - 4a + 1 \leq 0$, which is equivalent to

$$1 - \frac{1}{\sqrt{2}} \leq a \leq 1 + \frac{1}{\sqrt{2}}.$$

Note that

$$\begin{aligned}
P(0) &= \frac{(2a-1)^2}{2}, \\
P(1) &= 1 - 2a + b \\
&= 1 - 2a + \frac{(2a-1)^2}{2} \\
&= (1-2a)\left(\frac{3}{2} - a\right), \\
P(2) &= 4 - 4a + \frac{(2a-1)^2}{\sqrt{2}} \\
&= \frac{4a^2 - 12a + 9}{2} \\
&= \frac{(2a-3)^2}{2}.
\end{aligned}$$

This shows that a lies in

$$\left(1 - \frac{1}{\sqrt{2}}, 1 + \frac{1}{\sqrt{2}}\right) \setminus \left\{\frac{1}{2}, \frac{3}{2}\right\}.$$

Note that for any a lying in the above set, taking $b = (2a-1)^2/2$, it follows that $P(0), P(1), P(2)$ are nonzero, and form a geometric progression. This shows that the desired set is equal to

$$\left(1 - \frac{1}{\sqrt{2}}, 1 + \frac{1}{\sqrt{2}}\right) \setminus \left\{\frac{1}{2}, \frac{3}{2}\right\}.$$

■

References

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[**Kos14**] Thomas Koshy. *Pell and Pell-Lucas numbers with applications*. Springer, New York, 2014, pp. xxiv+431. isbn: 978-1-4614-8488-2; 978-1-4614-8489-9. doi: 10.1007/978-1-4614-8489-9. url: http://dx.doi.org/10.1007/978-1-4614-8489-9 (cited p. 3)

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