MTH102

Linear Algebra

2025–2026–I Semester

Problem Sets

05 October 2025

Chapter 1. Problem set 1

Exercise 1.1. Determine the elements of the following sets.
- $\{x \in \mathbb{N} : x^2 - 1 = 0\}$.
- $\{x \in \mathbb{Z} : x^2 - 1 = 0\}$.

Exercise 1.2. What does it mean to say that A is not a subset of B?

Exercise 1.3. Show that if A is a subset of B and B is a subset of C, then A is a subset of C.

Exercise 1.4. Show that no element n of $\mathbb{N}$ satisfies $n^4 - 5n^2 + 6 = 0$.

Exercise 1.5. What does it mean to say that two sets A, B are not equal?

Exercise 1.6. Consider the sets

$$\{1, 2, 3, 4\}, \{3, 4, 5, 6\}.$$

Determine the elements common to these sets.

Exercise 1.7. Determine the union of the sets

$$\{1, 2, 3, 4\}, \{1, 3, 5, 7\}.$$

Exercise 1.8. If A, B are sets, show that

$$A \subseteq A \cup B, B \subseteq A \cup B.$$

Exercise 1.9. If A, B are subsets of a set C, then $A \cup B$ is a subset of C.

Exercise 1.10. If A is a set, show that

$$A \cup A = A.$$

Exercise 1.11 (Commutative property). If A, B are sets, show that

$$A \cup B = B \cup A.$$

Exercise 1.12 (Associative property). If A, B, C are sets, show that

$$A \cup B \cup C = (A \cup B) \cup C = A \cup (B \cup C).$$

Exercise 1.13. Determine the intersection of the sets

$$\{1, 2, 3, 4\}, \{1, 3, 5, 7\}.$$

Exercise 1.14. If A, B are sets, show that

$$A \cap B \subseteq A, A \cap B \subseteq B.$$

Exercise 1.15. If A, B, C are sets satisfying

$$C \subseteq A, C \subseteq B,$$

then show that

$$C \subseteq A \cap B$$

holds.

Exercise 1.16. Identify the integers among

$$1, 2, 3, 4, \ldots, 20$$

which are divisible by at least one of 2 and 5.

Exercise 1.17 (Inclusion-exclusion principle). Show that for finite subsets A, B, C of a set X,

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|.$$

Exercise 1.18. Determine the number of integers among

$$1, 2, 3, 4, \ldots, 100,$$

which are divisible by at least one of $6, 10, 15$.

Exercise 1.19. If A is a set, show that

$$A \cap A = A.$$

Exercise 1.20 (Commutative property). If A, B are sets, show that

$$A \cap B = B \cap A.$$

Exercise 1.21 (Associative property). If A, B, C are sets, show that

$$A \cap B \cap C = (A \cap B) \cap C = A \cap (B \cap C).$$

Exercise 1.22. Show that

$$A^c = \{x \in X : x \notin A\},$$

where X denotes the underlying universal set.

Exercise 1.23. Determine the complement of $\{1, 2, 3, 4\}$ in $\{1, 3, 5, 7\}$, and the complement of $\{1, 3, 5, 7\}$ in $\{1, 2, 3, 4\}$.

Exercise 1.24. If A, B are subsets of a set X, show that

$$A^c \cap B = B \setminus A.$$

Exercise 1.25. If A is a subset of a set X, then show that
$$A \cup A^c = X, A \cap A^c = \emptyset, (A^c)^c = A.$$

Exercise 1.26. If A, B are subsets of a set X, then show that
$$(A \cup B)^c = A^c \cap B^c, (A \cap B)^c = A^c \cup B^c.$$

Exercise 1.27. Let A, B be subsets of a set X. Show that the following statements are equivalent.
$$A \subseteq B,$$
$$A \cap B = A,$$
$$A \cup B = B,$$
$$B^c \subseteq A^c.$$

Exercise 1.28. Let A, B be subsets of a set X. Show that the sets
$$A \setminus B, B \setminus A$$
are disjoint.

Exercise 1.29. Determine the symmetric difference of $\{1, 2, 3, 4\}$ and $\{1, 3, 5, 7\}$.

Exercise 1.30. If A, B are sets, then show that
$$A \Delta B = (A \setminus B) \cup (B \setminus A),$$
$$A \Delta B = B \Delta A,$$
$$A \cup B = (A \Delta B) \cup (A \cap B),$$
$$(A \Delta B) \cap (A \cap B) = \emptyset.$$

Exercise 1.31. Write down the power set of each of the following sets:
$$\{1, 2\}, \{1, 2, 3\}, \{1, \{2, 3\}\}, \{1, 2, 3, 4\}.$$

Exercise 1.32. If A, B are finite sets, then show that
$$|A \times B| = |A| \times |B|,$$
where for a set X, its number of elements is denoted by $|X|$.

Exercise 1.33. Determine the cartesian product of $[1, 2]$ and $[3, 4] \cup [5, 6]$.

Exercise 1.34. Identify the set
$$\left\{x \in \mathbb{R} \setminus \{0\} : x + \frac{1}{x} \geq 2\right\}.$$

Exercise 1.35. Let A, B be subsets of $\mathbb{R}$ defined by
$$A = \{x \in \mathbb{R} : x^2 \geq 0\},$$
$$B = \{x \in \mathbb{R} : x^3 \geq 0\}.$$
Determine the sets $A \cup B$, $A \cap B$, $A \setminus B$, $B \setminus A$.

Exercise 1.36. For a positive integer k, let A_k denote the set of integral multiples of k, that is,

$$A_k := \{r \in \mathbb{Z} : r = k\ell \text{ for some } \ell \in \mathbb{Z}\}.$$

Let m, n be positive integers. For each of the following statements, determine the equivalent conditions on the integers m, n.

1. $A_m \subseteq A_n$
2. $A_m \subsetneq A_n$
3. $A_m \not\subseteq A_n$
4. $A_m = A_n$
5. $A_m \cap A_n = \emptyset$
6. $A_m \setminus A_n \neq \emptyset$
7. $A_m \setminus A_n \neq \emptyset$ or $A_n \setminus A_m \neq \emptyset$
8. $A_m \setminus A_n \neq \emptyset$ and $A_n \setminus A_m \neq \emptyset$

Exercise 1.37. Show that $\mathbb{R}$ is a subset of $\mathbb{C}$.

Exercise 1.38. If z, w are complex numbers, then show that

1. $\overline{z + w} = \overline{z} + \overline{w}$,
2. $\overline{z \cdot w} = \overline{z} \cdot \overline{w}$,
3. $|z| \geq 0$, and $|z| = 0$ if and only if $z = 0$,
4. $|z \cdot w| = |z|\ |w|$.

Exercise 1.39. If z is a complex number, then show that

$$z \cdot \overline{z} = |z|^2.$$

Exercise 1.40. For any real number θ, show that

$$|\cos\theta + i \sin\theta| = 1.$$

Chapter 2. Induction principles

Exercise 2.1. If A is a nonempty subset of $\mathbb{N}$, and a_1, a_2 are least elements of A, then show that $a_1 = a_2$.

Exercise 2.2. Show that

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$$

for all $n \in \mathbb{N}$.

Exercise 2.3. Show that

$$\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \cdots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$$

for all $n \in \mathbb{N}$.

Exercise 2.4. Show that

$$\frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} + \cdots + \frac{1}{n(n+1)(n+2)} = \frac{n(n+3)}{4(n+1)(n+2)}$$

for all $n \in \mathbb{N}$.

Exercise 2.5. Show that

$$3 + 11 + \cdots + (8n - 5) = 4n^2 - n$$

for all $n \in \mathbb{N}$.

Exercise 2.6. Show that

$$1^2 + 3^2 + \cdots + (2n-1)^2 = \frac{4n^3 - n}{3}$$

for all $n \in \mathbb{N}$.

Exercise 2.7. Show that

$$1^2 - 2^2 + 3^2 - 4^2 + \cdots + (-1)^{n+1} n^2 = (-1)^{n+1} \frac{n(n+1)}{2}$$

for all $n \in \mathbb{N}$.

Exercise 2.8. Show that

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6},$$

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$$

hold for any positive integer n.

Exercise 2.9. Show that

$$\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{2n-1} - \frac{1}{2n}$$

for all $n \in \mathbb{N}$.

Exercise 2.10. Show that $n^3 + 5n$ is divisible by 6 for all $n \in \mathbb{N}$.

Exercise 2.11. Show that $5^{2n} - 1$ is divisible by 8 for all $n \in \mathbb{N}$.

Exercise 2.12. Show that $5^n - 4n - 1$ is divisible by 16 for all $n \in \mathbb{N}$.

Exercise 2.13. Show that $6^n - 5n - 1$ is divisible by 25 for all $n \in \mathbb{N}$.

Exercise 2.14. Show that $n^3 + (n+1)^3 + (n+2)^3$ is divisible by 9 for all $n \in \mathbb{N}$.

Exercise 2.15. Show that

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} > \sqrt{n}$$

for all $n \in \mathbb{N}$ satisfying $n > 1$.

Exercise 2.16. Show that $3^n \geq n^2$ for all $n \in \mathbb{N}$.

Exercise 2.17. Show that $n! > 2^n$ for all $n \in \mathbb{N}$ satisfying $n \geq 4$.

Here are a few exercises where strong induction is useful.

Exercise 2.18. Show that every positive integer greater than 1 is a prime or is a product of prime numbers.

Exercise 2.19. Let the integers $x_1, x_2, \ldots$ be defined by

$$\begin{aligned} x_1 &:= 1, \\ x_2 &:= 2, \\ x_{n+2} &:= \frac{1}{2}(x_n + x_{n+1}) \end{aligned}$$

for all $n \in \mathbb{N}$. Show that $1 \leq x_n \leq 2$ for all $n \in \mathbb{N}$.

Exercise 2.20. The Fibonacci sequence $F_0, F_1, F_2, \ldots$ is defined by

$$\begin{aligned} F_0 &:= 0, \\ F_1 &:= 1, \\ F_2 &:= 1, \\ F_{n+2} &:= F_n + F_{n+1} \end{aligned}$$

for all $n \in \mathbb{N}$. Show that

$$F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$$

for all $n \in \mathbb{N} \cup \{0\}$.

Exercise 2.21. Show that

$$F_0^2 + F_1^2 + F_2^2 + \cdots + F_n^2 = F_n F_{n+1}$$

for all $n \in \mathbb{N} \cup \{0\}$.

Chapter 3. Matrices

Exercise 3.1. If A, B, C are matrices of the same size, show that
$$A + (B + C) = (A + B) + C.$$

Exercise 3.2. If A, B are matrices of the same size, show that
$$A + B = B + A.$$

Exercise 3.3. If A is an $m \times n$ matrix, show that
$$A + 0_{m \times n} = 0_{m \times n} + A = A$$
holds, where $0_{m \times n}$ denotes the $m \times n$ zero matrix, that is, the $m \times n$ matrix whose entries are all zero.

Exercise 3.4. Show that if A is a matrix and c, d are real numbers, then
$$(c + d)A = cA + dA$$
and
$$c(dA) = (cd)A.$$

Exercise 3.5. If A is a matrix, what is $\left(A^T\right)^T$?

Exercise 3.6. Show that if A, B are matrices of the same size, then
$$(A + B)^T = A^T + B^T.$$

Exercise 3.7. Show that if A denotes a matrix, then
$$\left(A^T\right)^T = A.$$

Exercise 3.8. Show that if A, B, C are matrices such that the products $A(BC)$ and $(AB)C$ are defined, then
$$A(BC) = (AB)C.$$

Exercise 3.9. Provide examples to show that in general, matrix multiplication is not commutative, that is, $AB \neq BA$ for some matrices A, B.

Exercise 3.10 (**). Show that for any $n \in \mathbb{N}$ with $n \geq 2$, there are $n \times n$ matrices A, B such that
$$AB \neq BA.$$

> **? Question**
>
> Can induction be used for the above exercise?

Exercise 3.11. If A is an $n \times n$ matrix, show that
$$AI_n = I_nA = A$$
holds, where I_n denotes the $n \times n$ diagonal matrix, with all diagonal entries equal to 1, that is,

$$I_n = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}.$$

Exercise 3.12. If A is an $m \times n$ matrix, show that

$$I_m A = A, A I_n = A$$

where I_m (resp. I_n) denote the $m \times m$ (resp. $n \times n$) identity matrix.

Exercise 3.13. If A is an $m \times n$ matrix with entries in $\mathbb{R}$ and c is a real number, then

$$cA = (cI_n)A = A(cI_m),$$

where I_n (resp. I_m) denotes the $n \times n$ (resp. $m \times m$) identity matrix.

Exercise 3.14. Show that if A, B are matrices, then

$$(AB)^T = B^T A^T.$$

Exercise 3.15. Show that if

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix},$$

then

$$A^n = \begin{pmatrix} F_{n-1} & F_n \\ F_n & F_{n+1} \end{pmatrix}$$

for all $n \in \mathbb{N}$ with $n \geq 1$, where F_n denotes the n-th Fibonacci number.

Exercise 3.16. Compute

$$\begin{pmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 1 & 0 \\ \vdots & ⋰ & \vdots & \vdots \\ 1 & \cdots & 0 & 0 \end{pmatrix}^2.$$

Exercise 3.17. Show that if A, B, C are matrices such that the sum $B + C$, and the products $AB, AC, A(B + C)$ are defined, then

$$A(B + C) = AB + AC.$$

Exercise 3.18. Show that if A, B, C are matrices such that the sum $B + C$, and the products $AB, AC, A(B + C)$ are defined, then

$$(A + B)C = AC + BC.$$

Exercise 3.19. If A is an $m \times n$ matrix, then show that for all $i = 1, 2, \dots, n$, the product Ae_i is equal to the i-th column of A, where e_i denotes the i-th standard basis vector of $\mathbb{R}^n$, that is,

$$e_i = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}$$

with 1 in the i-th position, and 0's elsewhere.

Exercise 3.20. Let A, B be invertible $n \times n$ matrices. Show that AB is invertible, and that

$$(AB)^{-1} = B^{-1}A^{-1}.$$

Compare the above exercise with Exercise 3.14.

Exercise 3.21. If A, B are 2×2 matrices, then show that

$$\det(AB) = \det(A)\det(B).$$

Exercise 3.22. Solve the system of equations

$$\begin{aligned} 5x + 2y &= 11, \\ 3x + 4y &= 8. \end{aligned}$$

Exercise 3.23. Solve the system of equations

$$\begin{aligned} 12x - 25y &= -47, \\ -7x + 30y &= 51. \end{aligned}$$

Exercise 3.24. Solve the system of equations

$$\begin{aligned} 2x + 3y + z &= 1, \\ 4x + y + 2z &= 2, \\ 3x + 2y + 3z &= 3. \end{aligned}$$

Exercise 3.25. Let A, B be 2×2 matrices. Show that if AB is invertible, then both A and B are invertible.

Compare the above with Exercise 3.61, Exercise 3.62, Exercise 3.73.

Exercise 3.26. Let A be an $n \times n$ matrix. If P is an invertible $n \times n$ matrix, then show that

$$\left(PAP^{-1}\right)^k = PA^kP^{-1}$$

holds for any positive integer k.

Exercise 3.27. Let A be a 2×2 matrix. Let V be the set of solutions to the system of equations

$$A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

that is,

$$V = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 : A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}.$$

Show that for any $u, v \in V$ and $c \in \mathbb{R}$, the element $u + v$ of $\mathbb{R}^2$ lies in V, and the element cu also lies in V.

Exercise 3.28. Let A be a 2×2 matrix. Let V be the set of vectors which can be expressed as a linear combination of the columns of A, that is[1],

$$V = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 : \begin{pmatrix} x \\ y \end{pmatrix} = A \begin{pmatrix} s \\ t \end{pmatrix} \text{ for some } s, t \in \mathbb{R} \right\}.$$

Show that for any $u, v \in V$ and $c \in \mathbb{R}$, the element $u + v$ of $\mathbb{R}^2$ lies in V, and the element cu also lies in V.

Compare the following exercise with Exercise 3.55.

Exercise 3.29. Let A be an $m \times n$ matrix. Let V be the set of solutions to the system of equations

$$Au = 0,$$

that is,

$$V = \{v \in \mathbb{R}^n : Av = 0\}.$$

Show that for any $u, v \in V$ and $c, d \in \mathbb{R}$, the element $cu + dv$ of $\mathbb{R}^m$ lies in V.

Compare the following exercise with Exercise 3.56.

Exercise 3.30. Let A be an $m \times n$ matrix. Let V be the set of vectors which can be expressed as a linear combination of the columns of A, that is,

$$V = \{v \in \mathbb{R}^m : v = Au \text{ for some } u \in \mathbb{R}^n\}.$$

Show that for any $u, v \in V$ and $c, d \in \mathbb{R}$, the element $cu + dv$ of $\mathbb{R}^m$ lies in V.

Exercise 3.31. Let A be a square matrix. Show that if A is symmetric and skew-symmetric, then A is the zero matrix.

Exercise 3.32. Show that for any $n \geq 2$, there are infinitely many $n \times n$ orthogonal matrices.

Exercise 3.33. Let A be an orthogonal matrix. Show that A is invertible and that

$$A^{-1} = A^T.$$

Exercise 3.34. Let A, B be orthogonal matrices of the same size. Show that the matrix product AB is also an orthogonal matrix.

Exercise 3.35. Let A, B be square matrices with complex entries. Show that

$$\overline{AB} = \overline{A}\,\overline{B}.$$

Exercise 3.36. Let A be a square matrix with complex entries. Show that

$$\overline{A^T} = \left(\overline{A}\right)^T.$$

Exercise 3.37. Let A, B be square matrices of the same size with complex entries. Show that

$$(A + B)^* = A^* + B^*.$$

[1] Observe that

$$A \begin{pmatrix} s \\ t \end{pmatrix} = sC_1 + tC_2,$$

where C_1, C_2 denote the first and second columns of A respectively.

Exercise 3.38. Let A be a matrix with complex entries. Show that

$$(A^*)^* = A.$$

Exercise 3.39. Let A, B be matrices with complex entries. Show that

$$(AB)^* = B^* A^*.$$

Exercise 3.40. Let A be an $n \times n$ hermitian matrix. Show that all the diagonal entries of A are real numbers. More generally, show that for any $1 \leq i, j \leq n$, the (i, j)-entry and the (j, i)-entry of A are complex conjugates of each other.

Exercise 3.41. Let A, B be hermitian matrices of the same size. Show that the matrix $A + B$ is also a hermitian matrix. Is the matrix AB also a hermitian matrix?

Exercise 3.42. State and prove an analogue of Exercise 3.40 for skew-hermitian matrices.

Exercise 3.43. Let A, B be skew-hermitian matrices of the same size. Show that the matrix $A + B$ is also a skew-hermitian matrix. Is the matrix AB also a skew-hermitian matrix?

Exercise 3.44. Let A be a square matrix with complex entries. Show that if A is hermitian and skew-hermitian, then A is the zero matrix.

Exercise 3.45. Let A be a unitary matrix. Show that A is invertible and that

$$A^{-1} = A^*.$$

Exercise 3.46. Let A, B be unitary matrices of the same size. Show that the matrix product AB is also a unitary matrix.

Exercise 3.47. Show that any orthogonal matrix is a unitary matrix. Is the converse true?

Exercise 3.48. Determine whether the following matrices are unitary, hermitian, both, or neither.

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Exercise 3.49. Let z, w be complex numbers such that $|z|^2 + |w|^2 = 1$. Show that

$$\begin{pmatrix} z & -\overline{w} \\ w & \overline{z} \end{pmatrix}$$

is a unitary matrix.

Exercise 3.50. Let A be a diagonal matrix with complex entries. Show that A is a unitary matrix if and only if all the diagonal entries of A have absolute value equal to 1.

Exercise 3.51. Show that any unitary matrix is a normal matrix. Is the converse true?

Exercise 3.52. Show that any hermitian matrix is a normal matrix. Is the converse true?

Exercise 3.53. Show that any skew-hermitian matrix is a normal matrix. Is the converse true?

Exercise 3.54. Let e_{ij}, e_{kl} be matrix units of the same size. Show that $e_{ij}e_{kl}$ is equal to the matrix unit e_{il} if $j = k$, and is equal to the zero matrix if $j \neq k$, that is,

$$e_{ij}e_{kl} = \begin{cases} e_{il} & \text{if } j = k, \\ 0 & \text{if } j \neq k. \end{cases}$$

Denote the set of $m \times n$ matrices with entries from $\mathbb{R}$ by $M_{m,n}(\mathbb{R})$.

Compare the following exercise with Exercise 3.29.

Exercise 3.55. Let A be an $m \times n$ matrix. Let V be the set of solutions to the system of equations

$$uA = 0,$$

that is,

$$V = \{v \in M_{1,m}(\mathbb{R}) : vA = 0\}.$$

Show that for any $u, v \in V$ and $c, d \in \mathbb{R}$, the element $cu + dv$ of $\mathbb{R}^m$ lies in V.

Compare the following exercise with Exercise 3.30.

Exercise 3.56. Let A be an $m \times n$ matrix. Let V be the set of row vectors which can be expressed as a linear combination of the rows of A, that is,

$$V = \{v \in M_{1,n}(\mathbb{R}) : v = u^T A \text{ for some } u \in \mathbb{R}^m\}.$$

Show that for any $u, v \in V$ and $c, d \in \mathbb{R}$, the element $cu + dv$ of $M_{1,n}(\mathbb{R})$ lies in V.

Exercise 3.57. Solve the system of linear equations in five variables x_1, x_2, x_3, x_4, x_5:

$$\begin{aligned} x_1 + 2x_2 - x_3 + 4x_4 + x_5 &= 7, \\ 2x_1 + 3x_2 + x_3 + 5x_4 + 2x_5 &= 14, \\ -x_1 + 4x_2 - 2x_3 - 3x_4 + x_5 &= -10. \end{aligned} \tag{1}$$

Exercise 3.58. Solve the system of equations in the variables x_1, x_2, x_3:

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 1, \\ 2x_1 + 3x_2 + x_3 &= 2, \\ -x_1 + 4x_2 - 2x_3 &= -3, \\ 3x_1 - x_2 + 4x_3 &= 4, \\ 5x_1 + 2x_2 + 3x_3 &= 5. \end{aligned}$$

Exercise 3.59. Solve the system of equations in the variables x_1, x_2, x_3, x_4:

$$\begin{aligned} x_1 + 2x_2 - x_3 + 4x_4 &= 5, \\ 2x_1 + 3x_2 + x_3 + 5x_4 &= 8, \\ -x_1 + 4x_2 - 2x_3 - 3x_4 &= -4, \\ 3x_1 - x_2 + 4x_3 + 2x_4 &= 7. \end{aligned}$$

Exercise 3.60. Solve the system of equations in the variables x_1, x_2, x_3:

$$\begin{aligned} x_1 + 2x_2 - 5x_3 &= 20, \\ 2x_1 + 5x_2 - 7x_3 &= 33, \\ -x_1 - 2x_2 + 4x_3 &= -17. \end{aligned}$$

Exercise 3.61. Let A, B be square matrices of the same size. Suppose that A is invertible and $AB = I$ holds. Show that B is invertible and $B = A^{-1}$.

Exercise 3.62. Let A, B be square matrices of the same size. Suppose that A is invertible and $BA = I$ holds. Show that B is invertible and $B = A^{-1}$.

Compare the above two exercises with Exercise 3.25, Exercise 3.73.

Exercise 3.63. Let A, B be matrices, suppose that the number of columns of A is equal to the number of rows of B. Suppose AB has at least two columns. Let C be the matrix obtained from AB by removing the last column of AB. Show that there exists a matrix B' such that $C = AB'$.

Exercise 3.64. Let A, B be matrices, suppose that the number of columns of A is equal to the number of rows of B. Suppose AB has at least two rows. Let C be the matrix obtained from AB by removing the last row of AB. Show that there exists a matrix A' such that $C = A'B$.

Exercise 3.65. Use elementary row operations to reduce the matrix

$$A = \begin{pmatrix} 1 & 2 & -5 \\ 2 & 5 & -7 \\ -1 & -2 & 4 \end{pmatrix}$$

in its row echelon form. Use the row echelon form to determine whether A is invertible. If A is invertible, use elementary matrices corresponding to the performed elementary row operations to find A^{-1}.

Exercise 3.66. Solve the system of equations in the variables x_1, x_2:

$$\begin{aligned} 2x_1 - 3x_2 &= 7, \\ -4x_1 + 5x_2 &= -11. \end{aligned}$$

Exercise 3.67. Transform the matrix

$$\begin{pmatrix} 0 & 2 & -3 & 0 & 0 \\ 1 & -1 & 4 & 0 & 5 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 3 & -5 & 0 & 1 \\ 0 & 3 & -5 & -1 & 2 \end{pmatrix}$$

into a matrix in row echelon form using elementary row operations.

Exercise 3.68. Solve the system of equations in the variables x_1, x_2, x_3, x_4:

$$\begin{aligned} 2x_2 - 3x_3 &= 0, \\ x_1 - x_2 + 4x_3 &= 5, \\ x_4 &= -2, \\ 3x_2 - 5x_3 &= 1, \\ 3x_2 - 5x_3 - x_4 &= 2. \end{aligned}$$

Exercise 3.69. Find the rank of the matrix

$$\begin{pmatrix} 1 & 2 & -1 & 1 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 1 & -\frac{2}{15} \\ 0 & 0 & 1 & -\frac{1}{14} \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Exercise 3.70. If A is an $m \times n$ matrix, then show that

$$\mathrm{rk}(A) \leq \min\{m, n\}.$$

Exercise 3.71. If A is an $n \times n$ matrix, then show that the following statements are equivalent.

1. The matrix A is invertible.
2. The rank of A is n.

Exercise 3.72. If A is an invertible square matrix, then show that A^{-1} is also invertible, and that

$$(A^{-1})^{-1} = A.$$

Exercise 3.73. If A, B are square matrices of the same size satisfying $BA = I$, then show that the matrices A, B are invertible, and are the inverses of each other.

Compare the above with Exercise 3.61, Exercise 3.62, Exercise 3.25.

Exercise 3.74. Solve the system of equations in the variables x, y, z:

$$x + 2y + 3z = 1,$$
$$4x + 5y + 6z = 2,$$
$$7x + 8y + 10z = 3.$$

Exercise 3.75. Show that the determinant of the identity matrix I_3 is 1, and that

$$\mathrm{adj}(I_3) = I_3.$$

Exercise 3.76. Let

$$A = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix},$$

where a, b, c are scalars. Show that $\det(A) = abc$, and that

$$\mathrm{adj}(A) = \begin{pmatrix} bc & 0 & 0 \\ 0 & ca & 0 \\ 0 & 0 & ab \end{pmatrix}.$$

Exercise 3.77. Let

$$A = \begin{pmatrix} 1 & -5 & 6 \\ -4 & 8 & -9 \\ 7 & 2 & 10 \end{pmatrix}.$$

Find $\det(A)$ and $\mathrm{adj}(A)$.

Exercise 3.78. Let

$$A = \begin{pmatrix} 2 & 1 & -1 \\ 3 & 2 & 1 \\ 4 & 1 & 2 \end{pmatrix}.$$

Find $\det(A)$ and $\operatorname{adj}(A)$.

Exercise 3.79. Let

$$A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

where a, b are scalars. Show that $\det(A) = ab$, and that

$$\operatorname{adj}(A) = \begin{pmatrix} b & 0 \\ 0 & a \end{pmatrix}.$$

Exercise 3.80. Compute the determinant of the matrix

$$A = \begin{pmatrix} 1 & -5 & 6 \\ 2 & 0 & 9 \\ -1 & 2 & -1 \end{pmatrix}$$

to determine whether it is invertible. If A is invertible, then find its inverse using its adjoint.

Exercise 3.81. Use Gaussian elimination to determine the row echelon form of the matrix

$$A = \begin{pmatrix} 1 & -5 & 6 \\ 2 & 0 & 9 \\ -1 & 2 & -1 \end{pmatrix}.$$

Use the row echelon form to determine whether A is invertible. If A is invertible, then find its inverse using elementary matrices, corresponding to the elementary row operations used to transform A into its row echelon form.

Exercise 3.82. Solve the following system of equations in the variables x, y, z.

$$x + 2y + 3z = 1,$$
$$4x - 5y + 6z = 2,$$
$$7x - y + 10z = 3.$$

Practice Problems

Chapter 4. Sets

Exercise 4.1. Let $A = \{1, 2, 3, 4\}$ and $B = \{3, 4, 5, 6\}$. Find
$$A \cap B, A \cup B, A \setminus B, B \setminus A, A \Delta B.$$

Exercise 4.2. Let $U = \{1, 2, ..., 10\}$ be the universal set. If $A = \{2, 4, 6, 8, 10\}$ and $B = \{1, 3, 5, 7, 9\}$, then find
$$A^c, B^c, (A \cup B)^c, (A \cap B)^c.$$

Exercise 4.3. Write down the power set of each of the following sets:
$$\{1, 2\}, \{1, 2, 3\}, \{1, 2, 3, 4\}.$$

Exercise 4.4. Identify the set
$$\left\{x \in \mathbb{R} \setminus \{0\} : x + \frac{1}{x} \geq 2\right\}.$$

Exercise 4.5. Let A, B be subsets of $\mathbb{R}$ defined by
$$A = \{x \in \mathbb{R} : x^2 \geq 0\},$$
$$B = \{x \in \mathbb{R} : x^3 \geq 0\}.$$
Determine the sets $A \cup B$, $A \cap B$, $A \setminus B$, $B \setminus A$.

Exercise 4.6. For a positive integer k, let A_k denote the set of integral multiples of k, that is,
$$A_k := \{r \in \mathbb{Z} : r = k\ell \text{ for some } \ell \in \mathbb{Z}\}.$$
Let m, n be positive integers. For each of the following statements, determine the equivalent conditions on the integers m, n.

1. $A_m \subseteq A_n$
2. $A_m \subsetneq A_n$
3. $A_m \not\subseteq A_n$
4. $A_m = A_n$

Chapter 5. Induction

Exercise 5.1. Show that
$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$$
for all $n \in \mathbb{N}$.

Exercise 5.2. Show that
$$\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \cdots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$$
for all $n \in \mathbb{N}$.

Exercise 5.3. Show that
$$\frac{1}{1\cdot 2\cdot 3}+\frac{1}{2\cdot 3\cdot 4}+\cdots+\frac{1}{n(n+1)(n+2)}=\frac{n(n+3)}{4(n+1)(n+2)}$$
for all $n \in \mathbb{N}$.

Exercise 5.4. Show that
$$3+11+\cdots+(8n-5)=4n^2-n$$
for all $n \in \mathbb{N}$.

Exercise 5.5. Show that
$$1^2+3^2+\cdots+(2n-1)^2=\frac{4n^3-n}{3}$$
for all $n \in \mathbb{N}$.

Exercise 5.6. Show that
$$1^2-2^2+3^2-4^2+\cdots+(-1)^{n+1}n^2=(-1)^{n+1}\frac{n(n+1)}{2}$$
for all $n \in \mathbb{N}$.

Exercise 5.7. Show that
$$1^2+2^2+3^2+\cdots+n^2=\frac{n(n+1)(2n+1)}{6},$$
$$1^3+2^3+3^3+\cdots+n^3=\left(\frac{n(n+1)}{2}\right)^2$$
hold for any positive integer n.

Exercise 5.8. Show that
$$\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots+\frac{1}{2n-1}-\frac{1}{2n}$$
for all $n \in \mathbb{N}$.

Exercise 5.9. Show that n^3+5n is divisible by 6 for all $n \in \mathbb{N}$.

Exercise 5.10. Show that $5^{2n}-1$ is divisible by 8 for all $n \in \mathbb{N}$.

Exercise 5.11. Show that 5^n-4n-1 is divisible by 16 for all $n \in \mathbb{N}$.

Exercise 5.12. Show that 6^n-5n-1 is divisible by 25 for all $n \in \mathbb{N}$.

Exercise 5.13. Show that $n^3+(n+1)^3+(n+2)^3$ is divisible by 9 for all $n \in \mathbb{N}$.

Exercise 5.14. Show that
$$\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\cdots+\frac{1}{\sqrt{n}}>\sqrt{n}$$
for all $n \in \mathbb{N}$ satisfying $n>1$.

Chapter 6. Matrices

Exercise 6.1. Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 5 & 1 \\ 1 & 6 \end{pmatrix}$. Compute $2A + 3B$, A^2B, AB^3, A^T.

Exercise 6.2. Suppose A is a 3×2 matrix, and

$$A\begin{pmatrix} 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ 8 \\ 14 \end{pmatrix},$$

$$A\begin{pmatrix} 3 \\ 7 \end{pmatrix} = \begin{pmatrix} -4 \\ 8 \\ 20 \end{pmatrix}.$$

Determine

$$A\begin{pmatrix} 103 \\ 407 \end{pmatrix}, A\begin{pmatrix} 97 \\ 393 \end{pmatrix}.$$

Exercise 6.3. Show that every vector in $\mathbb{R}^2$ can be expressed as a linear combination of the vectors

$$\begin{pmatrix} 1 \\ 4 \end{pmatrix}, \begin{pmatrix} 3 \\ 7 \end{pmatrix}.$$

Exercise 6.4. Write the following system of equations in matrix form $Ax = b$.

$$2x + 3y = 5,$$

$$4x - y = 1.$$

Determine if A is invertible. If it is, find A^{-1} and use it to solve the system.

Exercise 6.5. If A, B are 2×2 matrices, then show that

$$\det(AB) = \det(A)\det(B).$$

Exercise 6.6. Determine which of the following matrices are upper-triangular, lower-triangular, or neither.

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix}, \begin{pmatrix} 7 & 8 \\ 0 & 9 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{pmatrix}, \begin{pmatrix} 7 & 0 \\ 8 & 9 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 0 \\ 0 & 3 & 4 \\ 5 & 0 & 6 \end{pmatrix}.$$

Exercise 6.7. Determine which of the following matrices are symmetric, which are skew-symmetric, and which are neither.

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix}, \begin{pmatrix} 7 & 8 \\ 8 & 9 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 2 & 3 \\ -2 & 0 & 5 \\ -3 & -5 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 8 \\ -8 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

Exercise 6.8. Determine which of the following matrices are orthogonal.

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

Exercise 6.9. Let

$$A = \begin{pmatrix} 1 & -5 & 6 \\ -4 & 8 & -9 \\ 7 & 2 & 10 \end{pmatrix}.$$

Find $\det(A)$ and $\mathrm{adj}(A)$.

Exercise 6.10. Let

$$A = \begin{pmatrix} 2 & 1 & -1 \\ 3 & 2 & 1 \\ 4 & 1 & 2 \end{pmatrix}.$$

Find $\det(A)$ and $\mathrm{adj}(A)$.

Exercise 6.11. Let θ be a real number satisfying $0 < \theta < \frac{\pi}{2}$. Show that the matrix

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

is orthogonal.

Exercise 6.12. Let A be an orthogonal matrix. Show that A is invertible and that

$$A^{-1} = A^T.$$

Exercise 6.13. Write the following system of equations in matrix form $Ax = b$.

$$\begin{aligned} x_1 + 2x_2 - x_3 + 4x_4 &= 5, \\ 2x_1 + 3x_2 + x_3 + 5x_4 &= 8, \\ -x_1 + 4x_2 - 2x_3 - 3x_4 &= -4, \\ 3x_1 - x_2 + 4x_3 + 2x_4 &= 7. \end{aligned}$$

Perform Gaussian elimination on the augmented matrix $(A \mid b)$, to reduct it to row echelon form. Using this form, determine if the system has a solution. If it does, find all solutions of the above system in the variables x_1, x_2, x_3, x_4.

Exercise 6.14. Perform row reduction on the matrix

$$\begin{pmatrix} 1 & 2 & -5 \\ 2 & 5 & -7 \\ -1 & -2 & 4 \end{pmatrix}$$

to reduce it to row echelon form. Using these reductions, find the inverse of the above matrix, if it exists.

Exercise 6.15. Transform the matrix

$$\begin{pmatrix} 0 & 2 & -3 & 0 & 0 \\ 1 & -1 & 4 & 0 & 5 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 3 & -5 & 0 & 1 \\ 0 & 3 & -5 & -1 & 2 \end{pmatrix}$$

into a matrix in row echelon form using elementary row operations.

Exercise 6.16. Solve the system of equations in the variables x_1, x_2, x_3, x_4.

$$\begin{aligned} 2x_2 - 3x_3 &= 0, \\ x_1 - x_2 + 4x_3 &= 5, \\ x_4 &= -2, \\ 3x_2 - 5x_3 &= 1, \\ 3x_2 - 5x_3 - x_4 &= 2. \end{aligned}$$

Exercise 6.17. Let

$$A = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix},$$

where a, b, c are scalars. Show that $\det(A) = abc$, and that

$$\operatorname{adj}(A) = \begin{pmatrix} bc & 0 & 0 \\ 0 & ca & 0 \\ 0 & 0 & ab \end{pmatrix}.$$

Exercise 6.18. Compute the determinant of the matrix

$$A = \begin{pmatrix} 1 & -5 & 6 \\ 2 & 0 & 9 \\ -1 & 2 & -1 \end{pmatrix}$$

to determine whether it is invertible. If A is invertible, then find its inverse using its adjoint.

Exercise 6.19. Use Gaussian elimination to determine the row echelon form of the matrix

$$A = \begin{pmatrix} 1 & -5 & 6 \\ 2 & 0 & 9 \\ -1 & 2 & -1 \end{pmatrix}.$$

Use the row echelon form to determine whether A is invertible. If A is invertible, then find its inverse using elementary matrices, corresponding to the elementary row operations used to transform A into its row echelon form.

Exercise 6.20. Solve the following system of equations in the variables x, y, z.

$$\begin{aligned} x + 2y + 3z &= 1, \\ 4x - 5y + 6z &= 2, \\ 7x - y + 10z &= 3. \end{aligned}$$