Chapter 1. Sets

Exercise 1.1. Let $A = \{1, 2, 3, 4\}$ and $B = \{3, 4, 5, 6\}$. Find

$$A \cap B, A \cup B, A \setminus B, B \setminus A, A \Delta B.$$

Exercise 1.2. Let $U = \{1, 2, \ldots, 10\}$ be the universal set. If $A = \{2, 4, 6, 8, 10\}$ and $B = \{1, 3, 5, 7, 9\}$, then find

$$A^c, B^c, (A \cup B)^c, (A \cap B)^c.$$

Exercise 1.3. Write down the power set of each of the following sets:

$$\{1, 2\}, \{1, 2, 3\}, \{1, 2, 3, 4\}.$$

Exercise 1.4. Identify the set

$$\left\{x \in \mathbb{R} \setminus \{0\} : x + \frac{1}{x} \geq 2\right\}.$$

Exercise 1.5. Let A, B be subsets of $\mathbb{R}$ defined by

$$A = \{x \in \mathbb{R} : x^2 \geq 0\},$$

$$B = \{x \in \mathbb{R} : x^3 \geq 0\}.$$

Determine the sets $A \cup B$, $A \cap B$, $A \setminus B$, $B \setminus A$.

Exercise 1.6. For a positive integer k, let A_k denote the set of integral multiples of k, that is,

$$A_k := \{r \in \mathbb{Z} : r = k\ell \text{ for some } \ell \in \mathbb{Z}\}.$$

Let m, n be positive integers. For each of the following statements, determine the equivalent conditions on the integers m, n.

1. $A_m \subseteq A_n$
2. $A_m \subsetneq A_n$
3. $A_m \not\subseteq A_n$
4. $A_m = A_n$

Chapter 2. Induction

Exercise 2.1. Show that

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$$

for all $n \in \mathbb{N}$.

Exercise 2.2. Show that

$$\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \cdots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$$

for all $n \in \mathbb{N}$.

Exercise 2.3. Show that

$$\frac{1}{1\cdot 2\cdot 3}+\frac{1}{2\cdot 3\cdot 4}+\cdots+\frac{1}{n(n+1)(n+2)}=\frac{n(n+3)}{4(n+1)(n+2)}$$

for all $n \in \mathbb{N}$.

Exercise 2.4. Show that

$$3+11+\cdots+(8n-5)=4n^2-n$$

for all $n \in \mathbb{N}$.

Exercise 2.5. Show that

$$1^2+3^2+\cdots+(2n-1)^2=\frac{4n^3-n}{3}$$

for all $n \in \mathbb{N}$.

Exercise 2.6. Show that

$$1^2-2^2+3^2-4^2+\cdots+(-1)^{n+1}n^2=(-1)^{n+1}\frac{n(n+1)}{2}$$

for all $n \in \mathbb{N}$.

Exercise 2.7. Show that

$$1^2+2^2+3^2+\cdots+n^2=\frac{n(n+1)(2n+1)}{6},$$

$$1^3+2^3+3^3+\cdots+n^3=\left(\frac{n(n+1)}{2}\right)^2$$

hold for any positive integer n.

Exercise 2.8. Show that

$$\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots+\frac{1}{2n-1}-\frac{1}{2n}$$

for all $n \in \mathbb{N}$.

Exercise 2.9. Show that n^3+5n is divisible by 6 for all $n \in \mathbb{N}$.

Exercise 2.10. Show that $5^{2n}-1$ is divisible by 8 for all $n \in \mathbb{N}$.

Exercise 2.11. Show that 5^n-4n-1 is divisible by 16 for all $n \in \mathbb{N}$.

Exercise 2.12. Show that 6^n-5n-1 is divisible by 25 for all $n \in \mathbb{N}$.

Exercise 2.13. Show that $n^3+(n+1)^3+(n+2)^3$ is divisible by 9 for all $n \in \mathbb{N}$.

Exercise 2.14. Show that

$$\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\cdots+\frac{1}{\sqrt{n}}>\sqrt{n}$$

for all $n \in \mathbb{N}$ satisfying $n>1$.

Chapter 3. Matrices

Exercise 3.1. Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 5 & 1 \\ 1 & 6 \end{pmatrix}$. Compute $2A + 3B$, A^2B, AB^3, A^T.

Exercise 3.2. Suppose A is a 3×2 matrix, and

$$A\begin{pmatrix} 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ 8 \\ 14 \end{pmatrix},$$

$$A\begin{pmatrix} 3 \\ 7 \end{pmatrix} = \begin{pmatrix} -4 \\ 8 \\ 20 \end{pmatrix}.$$

Determine

$$A\begin{pmatrix} 103 \\ 407 \end{pmatrix}, A\begin{pmatrix} 97 \\ 393 \end{pmatrix}.$$

Exercise 3.3. Show that every vector in $\mathbb{R}^2$ can be expressed as a linear combination of the vectors

$$\begin{pmatrix} 1 \\ 4 \end{pmatrix}, \begin{pmatrix} 3 \\ 7 \end{pmatrix}.$$

Exercise 3.4. Write the following system of equations in matrix form $Ax = b$.

$$2x + 3y = 5,$$

$$4x - y = 1.$$

Determine if A is invertible. If it is, find A^{-1} and use it to solve the system.

Exercise 3.5. If A, B are 2×2 matrices, then show that

$$\det(AB) = \det(A)\det(B).$$

Exercise 3.6. Determine which of the following matrices are upper-triangular, lower-triangular, or neither.

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix}, \begin{pmatrix} 7 & 8 \\ 0 & 9 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{pmatrix}, \begin{pmatrix} 7 & 0 \\ 8 & 9 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 0 \\ 0 & 3 & 4 \\ 5 & 0 & 6 \end{pmatrix}.$$

Exercise 3.7. Determine which of the following matrices are symmetric, which are skew-symmetric, and which are neither.

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix}, \begin{pmatrix} 7 & 8 \\ 8 & 9 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 2 & 3 \\ -2 & 0 & 5 \\ -3 & -5 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 8 \\ -8 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

Exercise 3.8. Determine which of the following matrices are orthogonal.

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

Exercise 3.9. Let

$$A = \begin{pmatrix} 1 & -5 & 6 \\ -4 & 8 & -9 \\ 7 & 2 & 10 \end{pmatrix}.$$

Find $\det(A)$ and $\operatorname{adj}(A)$.

Exercise 3.10. Let

$$A = \begin{pmatrix} 2 & 1 & -1 \\ 3 & 2 & 1 \\ 4 & 1 & 2 \end{pmatrix}.$$

Find $\det(A)$ and $\operatorname{adj}(A)$.

Exercise 3.11. Let θ be a real number satisfying $0 < \theta < \frac{\pi}{2}$. Show that the matrix

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

is orthogonal.

Exercise 3.12. Let A be an orthogonal matrix. Show that A is invertible and that

$$A^{-1} = A^T.$$

Exercise 3.13. Write the following system of equations in matrix form $Ax = b$.

$$\begin{aligned} x_1 + 2x_2 - x_3 + 4x_4 &= 5, \\ 2x_1 + 3x_2 + x_3 + 5x_4 &= 8, \\ -x_1 + 4x_2 - 2x_3 - 3x_4 &= -4, \\ 3x_1 - x_2 + 4x_3 + 2x_4 &= 7. \end{aligned}$$

Perform Gaussian elimination on the augmented matrix $(A \mid b)$, to reduct it to row echelon form. Using this form, determine if the system has a solution. If it does, find all solutions of the above system in the variables x_1, x_2, x_3, x_4.

Exercise 3.14. Perform row reduction on the matrix

$$\begin{pmatrix} 1 & 2 & -5 \\ 2 & 5 & -7 \\ -1 & -2 & 4 \end{pmatrix}$$

to reduce it to row echelon form. Using these reductions, find the inverse of the above matrix, if it exists.

Exercise 3.15. Transform the matrix

$$\begin{pmatrix} 0 & 2 & -3 & 0 & 0 \\ 1 & -1 & 4 & 0 & 5 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 3 & -5 & 0 & 1 \\ 0 & 3 & -5 & -1 & 2 \end{pmatrix}$$

into a matrix in row echelon form using elementary row operations.

Exercise 3.16. Solve the system of equations in the variables x_1, x_2, x_3, x_4.

$$\begin{aligned} 2x_2 - 3x_3 &= 0, \\ x_1 - x_2 + 4x_3 &= 5, \\ x_4 &= -2, \\ 3x_2 - 5x_3 &= 1, \\ 3x_2 - 5x_3 - x_4 &= 2. \end{aligned}$$

Exercise 3.17. Let

$$A = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix},$$

where a, b, c are scalars. Show that $\det(A) = abc$, and that

$$\operatorname{adj}(A) = \begin{pmatrix} bc & 0 & 0 \\ 0 & ca & 0 \\ 0 & 0 & ab \end{pmatrix}.$$

Exercise 3.18. Compute the determinant of the matrix

$$A = \begin{pmatrix} 1 & -5 & 6 \\ 2 & 0 & 9 \\ -1 & 2 & -1 \end{pmatrix}$$

to determine whether it is invertible. If A is invertible, then find its inverse using its adjoint.

Exercise 3.19. Use Gaussian elimination to determine the row echelon form of the matrix

$$A = \begin{pmatrix} 1 & -5 & 6 \\ 2 & 0 & 9 \\ -1 & 2 & -1 \end{pmatrix}.$$

Use the row echelon form to determine whether A is invertible. If A is invertible, then find its inverse using elementary matrices, corresponding to the elementary row operations used to transform A into its row echelon form.

Exercise 3.20. Solve the following system of equations in the variables x, y, z.

$$\begin{aligned} x + 2y + 3z &= 1, \\ 4x - 5y + 6z &= 2, \\ 7x - y + 10z &= 3. \end{aligned}$$