MOPSS

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Mathematics Olympiad

Problem Solving Sessions

Department of Mathematics
IISER Bhopal

https://jpsaha.github.io/MOTP/MOPSS/

Suggested readings

- Evan Chen's advice On reading solutions, available at https://blog.evanchen.cc/2017/03/06/on-reading-solutions/.
- Evan Chen's Advice for writing proofs/Remarks on English, available at https://web.evanchen.cc/handouts/english/english.pdf.
- Notes on proofs by Evan Chen from OTIS Excerpts [Che25, Chapter 1].
- Tips for writing up solutions by Edward Barbeau, available at https://www.math.utoronto.ca/barbeau/writingup.pdf.
- Evan Chen discusses why math olympiads are a valuable experience for high schoolers in the post on Lessons from math olympiads, available at https://blog.evanchen.cc/2018/01/05/lessons-from-math-olympiads/.

List of problems and examples

§1

Exercise 1.1 (Brazil National Olympiad 2020 Level 3 P1, AoPS)**.** Prove that there are positive integers $a_1, a_2, \dots, a_{2020}$ such that

$$\frac{1}{a_1}+\frac{1}{2a_2}+\frac{1}{3a_3}+\cdots+\frac{1}{2020a_{2020}}=1.$$

Walkthrough —

(a) Show that for any positive integer n, there exist positive integers $a_1, a_2, \dots, a_n$ such that
$$\frac{1}{a_1}+\frac{1}{2a_2}+\frac{1}{3a_3}+\cdots+\frac{1}{na_n}=1.$$

(b) Observe that $1=\frac{1}{2}+\frac{1}{2}$, $1=\frac{1}{2}+\frac{1}{3}+\frac{1}{6}$, $1=\frac{1}{2}+\frac{1}{6}+\frac{1}{4}+\frac{1}{12}$ etc. hold.

(c) Also note that
$$\frac{1}{n}=\frac{1}{n+1}+\frac{1}{n(n+1)}.$$

Solution 1. We claim that for any positive integer n, there exist positive integers $a_1, a_2, \dots, a_n$ such that

$$\frac{1}{a_1}+\frac{1}{2a_2}+\frac{1}{3a_3}+\cdots+\frac{1}{na_n}=1.$$

We will prove this claim by induction on n. The base case $n=1$ is trivial, since we can take $a_1=1$. Also note that the case $n=2$ holds, since we can take $a_1=2$ and $a_2=1$. Assume now that the claim holds for some $n\geq 2$. Hence, there exist positive integers $a_1, a_2, \dots, a_n$ such that

$$\frac{1}{a_1}+\frac{1}{2a_2}+\frac{1}{3a_3}+\cdots+\frac{1}{na_n}=1.$$

Using

$$\frac{1}{n}=\frac{1}{n+1}+\frac{1}{n(n+1)},$$

we obtain

$$1=\frac{1}{a_1}+\frac{1}{2a_2}+\frac{1}{3a_3}+\cdots+\frac{1}{na_n}$$

$$= \frac{1}{a_1} + \frac{1}{2a_2} + \frac{1}{3a_3} + \cdots + \frac{1}{n(n+1)a_n} + \frac{1}{(n+1)a_n}.$$

This shows that the claim also holds for $n+1$. By the principle of mathematical induction, the claim holds for all positive integers n. In particular, the claim holds for $n = 2020$, which completes the proof. ■

Solution 2. Note that

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n(n+1)} = 1 - \frac{1}{n+1}$$

holds for all positive integers n. Taking

$$a_1 = n+1, a_2 = 1, a_3 = 2, \ldots, a_{n+1} = n$$

yields

$$\frac{1}{a_1} + \frac{1}{2a_2} + \frac{1}{3a_3} + \cdots + \frac{1}{(n+1)a_{n+1}} = 1.$$

■

Exercise 1.2 (Mexico National Olympiad 2013 P1, AoPS)**.** All prime numbers are written in order, $p_1 = 2, p_2 = 3, p_3 = 5, \ldots$. Find all pairs of positive integers a and b with $a - b \geq 2$, such that $p_a - p_b$ divides $2(a-b)$.

> **Walkthrough** —
>
> **(a)**

Solution 3. Let a, b be positive integers such that $a - b \geq 2$ and $p_a - p_b$ divides $2(a-b)$.

Let us consider the case that $b \geq 2$. Note that p_i, p_{i+1} differ by at least 2 for all $i \geq 2$, since all primes greater than 2 are odd. This gives

$$p_a - p_b = (p_a - p_{a-1}) + (p_{a-1} - p_{a-2}) + \cdots + (p_{b+1} - p_b) \geq 2(a-b).$$

Since $p_a - p_b$ divides $2(a-b)$, we must have $p_a - p_b = 2(a-b)$, implying that p_i, p_{i+1} differ by exactly 2 for all $i = b, b+1, \ldots, a-1$. Since $a \geq b+2$, it follows that the primes p_b, p_{b+1}, p_{b+2} are three consecutive primes differing by 2. Note that p_b, p_{b+1}, p_{b+2} are congruent to $0, 1, 2$ modulo 3 in some order. Thus, one of these three primes is divisible by 3. We obtain $p_b = 3$, which gives $b = 2$. If $a \geq b+3$, then $p_{b+3} - p_{b+2} \geq 4$ holds, which implies

$$p_a - p_b = (p_a - p_{a-1}) + (p_{a-1} - p_{a-2}) + \cdots + (p_{b+1} - p_b) \geq 2(a-b) + 2.$$

This contradicts the fact that $p_a - p_b \leq 2(a-b)$. It follows that $a \leq b+2$. Also note that $a \geq b+2$. Thus, we conclude that $a = b+2 = 4$.

It remains to consider the case $b = 1$, which we assume from now on. We have $p_a - p_1 = p_a - 2$ divides $2(a-1)$. Since p_a is odd, it follows that $p_a - 2$ is odd, and hence it divides $a-1$. This implies that $p_a - 2 \leq a - 1$, or equivalently $p_a \leq a + 1$. Using $a \geq b + 2 = 3$, we obtain

$$p_a = (p_a - p_{a-1}) + \cdots + (p_3 - p_2) + p_2 \geq 2(a-2) + 3 = 2a - 1.$$

Combining the two inequalities, we get $2a - 1 \leq a + 1$, or equivalently $a \leq 2$, contradicting $a \geq 3$.

This proves that (a, b) is equal to $(4, 2)$. Note that this pair indeed satisfies the condition since $p_4 - p_2 = 7 - 3 = 4$ divides $2(4-2) = 4$. Consequently, the only solution is $(a, b) = (4, 2)$. $\blacksquare$

Exercise 1.3 (Brazil National Olympiad 2020 Level 2 P2, AoPS)**.** The following statement is written on a board:

> The equation $x^2 - 824x + \blacksquare 143 = 0$ has two integer solutions.

In the above, $\blacksquare$ represents the first few digits of a number that is blurred. What are the possible equations originally written on the board?

> **Walkthrough** —
>
> **(a)**

Solution 4. Let α be an integer root of the polynomial $x^2 - 824x + \blacksquare 143$. Note that

$$(\alpha - 412)^2 = 412^2 - \blacksquare 143$$

holds. Note that α is odd. Also note that

$$(\alpha - 412)^2 \equiv 1 \pmod{100}.$$

This implies that

$$|\alpha - 412|^2 \equiv 1 \pmod{25},$$

which yields

$$(|\alpha - 412| - 1)(|\alpha - 412| + 1) \equiv 0 \pmod{25}.$$

Since not both of $|\alpha - 412| - 1$ and $|\alpha - 412| + 1$ are multiples of 5, it follows that $|\alpha - 412|$ is congruent to either 1 or -1 modulo 25. Since $\alpha - 412$ is odd, it follows that $|\alpha - 412|$ is congruent to either 1 or -1 modulo 50. Write

$$|\alpha - 412| = 50k \pm 1$$

for some integer $k \geq 0$. We obtain

$$(50k \pm 1)^2 = 412^2 - \blacksquare 143.$$

Note that

$$412^2 \equiv 800 \times 12 + 144 \equiv 744 \pmod{1000}$$

holds. This gives

$$|\alpha - 412|^2 \equiv 601 \pmod{1000},$$

and hence, we have

$$2500k^2 + 1 \pm 100k = 412^2 - \blacksquare 143.$$

This implies that

$$500k^2 + 1 \pm 100k \equiv 744 - 143 \pmod{1000},$$

which yields

$$500k^2 \pm 100k \equiv 600 \pmod{1000},$$

which is equivalent to

$$5k^2 \pm k \equiv 6 \pmod{10}.$$

This shows that

$$k \equiv \pm 1 \pmod{5}.$$

Since $|\alpha - 412| < 412$, it follows that

$$0 \leq k \leq 8.$$

This implies that $|\alpha - 412|$ is equal to one of

$$50 \pm 1, 50 \times 4 \pm 1, 50 \times 6 \pm 1,$$

that is, $|\alpha - 412|$ is one of

$$49, 51, 199, 201, 299, 301.$$

Note that

$$49^2 \equiv 401 \pmod{1000}, 201^2 \equiv 401 \pmod{1000}, 299^2 \equiv 401 \pmod{1000}.$$

This shows that $|\alpha - 412|$ is one of

$$51, 199, 301.$$

Using

$$\blacksquare 143 = 412^2 - (\alpha - 412)^2,$$

it follows that $\blacksquare 143$ is one of

$$167143, 130143, 79143.$$

This implies that $\blacksquare$ is one of

$$79, 130, 167.$$

Moreover, using

$$79143 = 412^2 - 301^2, 130143 = 412^2 - 199^2, 167143 = 412^2 - 51^2,$$

it follows that the equation $x^2 - 824x + a = 0$ has two integer solutions for any $a \in \{79143, 130143, 167143\}$.

We conclude that the possible values of ■ are $79, 130, 167$. Hence, the possible equations originally written on the board are

$$x^2 - 824x + 79143 = 0, \quad x^2 - 824x + 130143 = 0, \quad x^2 - 824x + 167143 = 0.$$

■

Exercise 1.4 (British Mathematical Olympiad Round 1 2008/9 P4)**.** Find all positive integers n such that both $n + 2008$ divides $n^2 + 2008$ and $n + 2009$ divides $n^2 + 2009$.

Walkthrough —

(a)

Solution 5. Let n be a positive integer such that both $n + 2008$ divides $n^2 + 2008$ and $n + 2009$ divides $n^2 + 2009$. Note that if a is a positive integer such that $n + a$ divides $n^2 + a$, then noting that

$$\begin{aligned} n^2 + a &= (n + a)^2 - 2na - a^2 + a \\ &= (n + a)^2 - 2a(n + a) + a^2 + a, \end{aligned}$$

it follows that $n + a$ divides $a^2 + a = a(a + 1)$. Thus, $n + 2008$ divides

$$2008 \cdot 2009 = 2^3 \cdot 7^2 \cdot 41 \cdot 251,$$

and $n + 2009$ divides

$$2009 \cdot 2010 = 2 \cdot 3 \cdot 5 \cdot 7^2 \cdot 41 \cdot 67.$$

Let us consider the case that 251 does not divide $n + 2008$. Since $n + 2008$ is greater than $2^3 \cdot 7^2$, it follows that 41 divides $n + 2008$. If 7 does not divide $n + 2008$, then $n + 2008$ divides $2^3 \cdot 41$, which is not possible since $n + 2008 \geq 2009 = 7^2 \cdot 41 > 2^3 \cdot 41$. Thus, 7 divides $n + 2008$. Since the integers $n + 2008, n + 2009$ have no factor in common other than 1, it follows that $n + 2009$ divides $2 \cdot 3 \cdot 5 \cdot 67 = 2010$, which implies that $n = 1$.

It remains to consider the case that 251 divides $n + 2008$, which we assume henceforth. Note that $n + 2009$ is not divisible by 4, and hence, $n \not\equiv 3 \pmod 4$. Let us consider the case that n is odd. Then, $n + 2008$ is odd, and hence, $n + 2008$ divides $7^2 \cdot 41 \cdot 251$. Using $n + 2008 \equiv 1 \pmod 4$, $251 \equiv 3 \pmod 4$

and $41 \equiv 1 \pmod 4$, it follows that 7 divides $n+2008$. Using $n+2008 \geq 2009$ and $n+2008 \equiv 1 \pmod 4$, we obtain that $n+2008$ is equal to none of

$$7 \cdot 251, 7^2 \cdot 251.$$

Hence, $n+2008$ is divisible by 7 and 41. Since the integers $n+2008, n+2009$ have no factor in common other than 1, it follows that $n+2009$ divides $2 \cdot 3 \cdot 5 \cdot 67 = 2010$, which implies that $n = 1$. It remains to consider the case that n is even, which we assume henceforth. Since $n+2008$ is greater than $2^3 \cdot 251$, it follows that $n+2008$ is divisible by at least one of $7, 41$. If each of $7, 41$ divides $n+2008$, then since the integers $n+2008, n+2009$ have no factor in common other than 1, it follows that $n+2009$ divides $3 \cdot 5 \cdot 67 = 1005$, which is impossible since $n+2009 \geq 2010$. Thus, exactly one of $7, 41$ divides $n+2008$.

Let us consider the case that 7 divides $n+2008$. Then $n+2009$ divides $3 \cdot 5 \cdot 41 \cdot 67$. Since $n+2009 \geq 2010$, it follows that 41 and 67 divide $n+2009$. Note that

$$41 \equiv -1 \pmod 7, \quad 67 \equiv -3 \pmod 7$$

holds, implying that $41 \cdot 67 \equiv 3 \pmod 7$. Since $n+2009 \equiv 1 \pmod 7$, we obtain $n+2009 = 5 \cdot 41 \cdot 67$, which yields

$$n+2008 \equiv 4 - 1 \equiv 0 \pmod 3.$$

Thus, $n+2008$ is divisible by 3, which is a contradiction. It remains to consider the case that 41 divides $n+2008$, which we assume henceforth. Note that $n+2009$ divides $3 \cdot 5 \cdot 7^2 \cdot 67$. Since $n+2009 \geq 2010$, it follows that 7 divides $n+2009$. This gives $n+2008 \equiv -1 \pmod 7$. Note that $41 \cdot 251 \equiv 41^1 \equiv 1 \pmod 7$ holds. Since $n+2008$ is a multiple of $2 \cdot 41 \cdot 251$, and divides $2^3 \cdot 41 \cdot 251$, it follows that one of $2, 2^2, 2^3$ is congruent to -1 modulo 7, which is a contradiction.

Therefore, there is only one positive integer n such that both $n+2008$ divides n^2+2008 and $n+2009$ divides n^2+2009, and that is $n = 1$. ■

References

[Che25] Evan Chen. *The OTIS Excerpts*. Available at https://web.evanchen.cc/excerpts.html. 2025, pp. vi+289 (cited p. 1)